

Bernays-Schönfinkel-Ramsey with Simple Bounds is NEXPTIME-complete*

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Abstract

Linear arithmetic extended with free predicate symbols is undecidable, in general. We show that the restriction of linear arithmetic inequations to simple bounds extended with the Bernays-Schönfinkel-Ramsey free first-order fragment is decidable and NEXPTIME-complete. The result is almost tight because the Bernays-Schönfinkel-Ramsey fragment is undecidable in combination with linear difference inequations, simple additive inequations, quotient inequations and multiplicative inequations.

1 Introduction

Satisfiability of the Bernays-Schönfinkel-Ramsey (BSR) fragment of first-order logic is decidable and NEXPTIME-complete [Lew80]. The complexity remains if the fragment is restricted to a clause normal form. Only further restrictions on the number of literals per clause enable better complexity results [Pla84]. Its extension with linear arithmetic is undecidable. For example, Halpern [Hal91] showed that the combination of Presburger Arithmetic with one unary predicate yields undecidability. His proof relies on $\forall\exists$ quantifier alternations on the arithmetic part. The naturals can be defined on the basis of linear real arithmetic with the help of one (extra) unary predicate. Fietzke and Weidenbach [FW12] showed undecidability for the combination of linear real arithmetic with several binary or one ternary predicate. The proof is based on a reduction of the two-counter machine model [Min67] to this fragment where a purely universal quantifier prefix suffices. Two-counter machine instructions are translated into clauses of the form

$$\begin{array}{ll} (i) & x' = x + 1 \parallel P(i, x, y) \rightarrow P(i + 1, x', y) \\ (ii) & x = 0 \parallel P(i, x, y) \rightarrow P(j, x', y) \\ & x > 0, x' = x - 1 \parallel P(i, x, y) \rightarrow P(i + 1, x', y) \end{array}$$

where $P(i, x, y)$ models the program at instruction i with counter values x, y . Then, clause (i) models the increment of counter x (analogous for y) and a go to the next instruction; clauses (ii) model the conditional decrement of x (analogous for y) and, otherwise, a jump to instruction j . The start state is represented by a clause $\parallel \rightarrow P(1, n, m)$ for two positive integer values n, m , and the halt instruction is represented by an atom $P(\text{halt}, x, y)$ and reachability of the halting state by a clause $\parallel P(\text{halt}, x, y) \rightarrow \square$. Then, a two-counter machine program halts if and only if the BSR clause set of linear arithmetic with one ternary predicate constructed out of the program is unsatisfiable. Note that for this reduction it does not matter whether integer or real arithmetic is the underlying arithmetic semantics. The first argument of P is always a natural.

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Our first contribution is refinements of the two-counter machine reduction where the arithmetic constraints are further restricted to linear difference inequations $x - y \triangleleft c$, simple additive inequations $x + y \triangleleft c$, quotient inequations $x \triangleleft c \cdot y$ and multiplicative inequations $x \cdot y \triangleleft c$ where $c \in \mathbb{R}$, and $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$. Under all these restrictions, the combination remains undecidable, respectively (see Section 7).

On the positive side, we prove decidability of the restriction to arithmetic constraints consisting of simple bounds of the form $x \triangleleft c$, where \triangleleft and c are as above. Underlying the result is the observation that similar to the finite model property of BSR, only finitely many test points are sufficient for the arithmetic simple bounds constraints. Our construction is motivated by results from quantifier elimination [LW93] and hierachic superposition [BGW94, KW12, FW12]. For example, consider the two clauses

$$\begin{array}{l} x_2 \neq 5 \parallel R(x_1) \rightarrow Q(u_1, x_2) \\ y_1 < 7, y_2 \leq 2 \parallel \quad \rightarrow Q(c, y_2), R(y_1) \end{array}$$

where u_1 is a free first-order variable, x_i, y_i are variables over the reals, and c is a free first-order constant. Our main result reveals that this clause set is satisfiable if and only if for every variable at the first argument of Q the constant c is substituted (Corollary 16), for the second argument of Q the abstract real values $5+\varepsilon$ and $-\infty$ and for R the value $-\infty$ (Definitions 6, 8, Lemma 13). The instantiation does not need to consider the simple bounds $y_1 < 7, y_2 \leq 2$, because it is sufficient to explore the reals either from $-\infty$ upwards or from $+\infty$ downwards, as is similarly done in linear quantifier elimination [LW93]. Also instantiation does not need to consider the value $5 + \varepsilon$ for R , motivated by the fact that hierachic superposition will not derive the respective simple bound for the first argument of R in any generated clause [BGW94]. This idea can be extended to free argument positions of predicates and the respective free constants. Every BSR clause set combined with simple bounds is always sufficiently complete because it does not contain a free non-constant function symbol.

All abstract values are represented by Skolem constants over the reals, together with defining axioms. For the example, we introduce the fresh Skolem constants $\alpha_{-\infty}$ to represent $-\infty$ and $\alpha_{5+\varepsilon}$ to represent $5 + \varepsilon$ together with axioms expressing $\alpha_{-\infty} < 2 < 5 < \alpha_{5+\varepsilon}$. Eventually, we obtain the ground clause set

$$\begin{array}{ll} \alpha_{-\infty} \geq 2 \parallel & \rightarrow \square \\ 5 \geq \alpha_{5+\varepsilon} \parallel & \rightarrow \square \\ \alpha_{5+\varepsilon} \neq 5 \parallel R(\alpha_{-\infty}) \rightarrow Q(c, \alpha_{5+\varepsilon}) \\ \alpha_{-\infty} \neq 5 \parallel R(\alpha_{-\infty}) \rightarrow Q(c, \alpha_{-\infty}) \\ \alpha_{-\infty} < 7, \alpha_{5+\varepsilon} \leq 2 \parallel & \rightarrow Q(c, \alpha_{5+\varepsilon}), R(\alpha_{-\infty}) \\ \alpha_{-\infty} < 7, \alpha_{-\infty} \leq 2 \parallel & \rightarrow Q(c, \alpha_{-\infty}), R(\alpha_{-\infty}) \end{array}$$

which has the model $\alpha_{-\infty}^A = 1, \alpha_{5+\varepsilon}^A = 6, R^A = \{1\}, Q^A = \{(c, 6), (c, 1)\}$, for instance. Using the result that every BSR clause set with simple bounds has a satisfiability-preserving ground instantiation with respect to finitely many constants, we prove NEXPTIME-completeness of the fragment in Section 5. For this result, the fine grained instantiation introduced in Section 3 and explained above by example is not needed. However, our further goal is to develop useful reasoning procedures for the fragment for which a smaller set of instances improves efficiency a lot. For the same reason, we do not restrict our attention to the BS fragment, but consider equality as a first class citizen of the logic from the very beginning.

Once a BSR clause set with simple bounds is grounded, there are a variety of efficient decision procedures known, such as SMT solvers employing the Nelson-Oppen [NO79] principle. However, for a large number of Skolem constants or large clause sets, an a-priori grounding may, due to its exponential increase in size, not be affordable. We are not aware of any calculus that is actually a decision procedure for the BSR fragment with simple bounds, although some work in this direction has been done already [BFT08, Rüm08, BLdM11, KW12].

The decidability result on simple bounds can be lifted to constraints of the form $x \triangleleft s$ where x is the only variable and s a ground expression. The lifting is done by the introduction of

further Skolem constants for complicated ground terms, following ideas of [KW12] and presented in Section 6. The paper ends with a conclusion, Section 8.

2 Basic Definitions

Hierarchic combinations of first-order logic with background theories [BGW94] build upon sorted logic with equality. A (*sorted*) *signature* Σ consists of a set Ξ of sorts, a set Ω of function symbols together with sort information and a set Π of predicate symbols also equipped with sort information. For the sake of simplicity, we restrict ourselves to two sorts: the *base sort* \mathcal{R} – interpreted as the set \mathbb{R} of reals by all the interpretations we shall consider – and the *free sort* \mathcal{S} – interpreted by some freely selectable nonempty domain. Throughout the paper we use convenient notation such as $P : \xi_1 \times \dots \times \xi_m \in \Pi$ to address an m -ary predicate symbol with full sort information; $P/m \in \Pi$ if there is some predicate symbol P of arity m in Π ; $P \in \Pi$ if there is some m such that $P/m \in \Pi$. To avoid confusion, we assume that Π contains at most one pair $P : \xi_1 \times \dots \times \xi_m$ for every P , and do not allow for multiple occurrences of P with different arity or sorting. Similar conventions shall hold for function symbols. In addition, we occasionally use the notation $\Omega \cap \mathcal{C}$ (where \mathcal{C} is an arbitrary set of constant symbols without annotated sorting information) to denote the set $\{c \mid c : \xi \in \Omega \text{ for some sort } \xi \text{ and } c \in \mathcal{C}\}$.

We instantiate the framework of hierarchic specification from [BGW94] by the hierarchic combination of the BSR fragment of first-order logic with a base theory allowing for the formulation of simple constraints on real-valued constant symbols and variables. Formally, the used *specification of the base theory* comprises the base signature $\Sigma_{\text{LA}} := \langle \{\mathcal{R}\}, \Omega_{\text{LA}} \cup \Omega_{\text{LA}}^\alpha, \Pi_{\text{LA}} \rangle$, where

$$\begin{aligned} \Omega_{\text{LA}} &:= \{r : \mathcal{R} \mid r \in \mathbb{R}\} \cup \{c_1 : \mathcal{R}, \dots, c_\kappa : \mathcal{R}\}, \\ \Omega_{\text{LA}}^\alpha &:= \{\alpha_{-\infty} : \mathcal{R}\} \cup \{\alpha_{c+\varepsilon} : \mathcal{R} \mid c \in \Omega_{\text{LA}}\}, \\ \Pi_{\text{LA}} &:= \{< : \mathcal{R}^2, \leq : \mathcal{R}^2, = : \mathcal{R}^2, \neq : \mathcal{R}^2, \geq : \mathcal{R}^2, > : \mathcal{R}^2\}, \end{aligned}$$

together with the class \mathfrak{M} of models containing one *base model* \mathcal{M} for every possible allocation of the Skolem constants c_1, \dots, c_κ in Ω_{LA} and the ones in $\Omega_{\text{LA}}^\alpha$ to real numbers. Moreover, each of the base models $\mathcal{M} \in \mathfrak{M}$ shall extend the *standard model* \mathcal{M}_{LA} of linear arithmetic over the reals, i.e. $\mathcal{R}^{\mathcal{M}} = \mathbb{R}$ and $c^{\mathcal{M}} = c$ for every constant symbol $c \in \Omega_{\text{LA}} \cap \mathbb{R}$ and $\triangleleft^{\mathcal{M}} = \triangleleft$ for every $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$ – in other words all real numbers serving as constant symbols represent their canonical value under each $\mathcal{M} \in \mathfrak{M}$ and every such \mathcal{M} interprets all predicate symbols $<, \leq, =, \neq, \geq, >$ by their standard meaning on the reals.

The definition of the base theory leaves the exact number κ of additional Skolem constants open; it may be 0. While adding all reals as constant symbols to the signature serves the aim of defining \mathfrak{M} to contain term-generated models only, the Skolem constants c_1, \dots, c_κ serve at least two purposes. Firstly, the modeling capabilities of our logical language are enhanced by real-valued constant symbols of which the exact value is *not* predetermined. Secondly, in Section 6 we outline a technique which allows us to soften our requirements towards the syntax a bit so that ground non-constant terms s of the base sort become admissible in addition to constant symbols and variables. The method has already been described in [KW12] under the name *basification* and introduces defining unit clauses such as $c = s$ where c is a fresh Skolem constant and s is a variable-free term.

In fact, we introduce even more Skolem constants by adding the set $\Omega_{\text{LA}}^\alpha$ to the base signature. For notational convenience, we syntactically distinguish this special kind of base-sort constant symbols α_t , t being of the form $-\infty$ or $d + \varepsilon$ for arbitrary base-sort constant symbols $d \in \Omega_{\text{LA}}$. These will play a key role when we instantiate base-sort variables later on. We associate an inherent meaning with constant symbols α_t that will be formalized by means of special of axioms.

We hierarchically extend the base specification $\langle \Sigma_{\text{LA}}, \mathfrak{M} \rangle$ by the free sort \mathcal{S} and finite sets Ω and Π of *free constant symbols* and *free predicate symbols*, respectively, each equipped with appropriate sort information. We use the symbol \approx to denote the built-in equality predicate on \mathcal{S} . To avoid confusion, we assume each constant or predicate symbol to occur in at most one of

the sets Ω_{LA} , $\Omega_{\text{LA}}^\alpha$, Ω , Π_{LA} , Π and that none of these sets contains the \approx symbol. In the light of sorted terms we consider two disjoint countably infinite sets of variables $V_{\mathcal{R}}$ – the *variables of the base sort*, usually denoted x, y, z – and $V_{\mathcal{S}}$ – the *free-sort variables*, usually denoted u, w .

Definition 1 (BSR Clause Fragment with Simple Bounds).

Let $\langle \{\mathcal{R}, \mathcal{S}\}, \Omega, \Pi \rangle$ be a signature such that Ω exclusively contains constant symbols c of the free sort and no function symbols of greater arity, and such that for every predicate symbol $P : \xi_1 \times \dots \times \xi_m \in \Pi$ and every argument position $i \leq m$ it holds $\xi_i \in \{\mathcal{R}, \mathcal{S}\}$.

An *atomic constraint* is of the form $c \triangleleft d$ or $x \triangleleft d$ or $\alpha_t \triangleleft d$ or $x = \alpha_t$ or $\alpha_t = \alpha_{t'}$ with $c, d \in \Omega_{\text{LA}}$, $x \in V_{\mathcal{R}}$, $\alpha_t, \alpha_{t'} \in \Omega_{\text{LA}}^\alpha$ and $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$.

A *free atom* A is either of the form $s \approx s'$ with s, s' being free-sort constant symbols in Ω or free-sort variables in $V_{\mathcal{S}}$, respectively, or A is of the form $P(s_1, \dots, s_m)$, where $P : \xi_1 \times \dots \times \xi_m \in \Pi$ is an m -ary predicate symbol. For each $i \leq m$ the term s_i shall be of the sort ξ_i . If $\xi_i = \mathcal{R}$, then s_i must be a variable $x \in V_{\mathcal{R}}$, and in case of $\xi_i = \mathcal{S}$, s_i may be a variable $u \in V_{\mathcal{S}}$ or a constant symbol $c : \mathcal{S} \in \Omega$.

A *clause* has the form $\Lambda \parallel \Gamma \rightarrow \Delta$, where Λ is a multiset of atomic constraints, and Γ and Δ are multisets of free atoms. We usually refer to Λ as the *constraint part* and to $\Gamma \rightarrow \Delta$ as the *free part* of the clause.

We conveniently denote the union of two multisets Θ and Θ' by juxtaposition Θ, Θ' . Moreover, we often write Θ, A as an abbreviation for the multiset union $\Theta \cup \{A\}$. In our clause notation empty multisets are usually omitted left of “ \rightarrow ” and denoted by \square right of “ \rightarrow ” (where \square at the same time stands for *falsity*).

Intuitively, clauses $\Lambda \parallel \Gamma \rightarrow \Delta$ can be read as $(\bigwedge \Lambda \wedge \bigwedge \Gamma) \rightarrow \bigvee \Delta$. Put into words, the multiset Λ stands for a conjunction of all atomic constraints it contains, Γ stands for a conjunction of the free atoms in it and Δ stands for a disjunction of the contained free atoms. All occurring variables are implicitly universally quantified. Requiring the free part $\Gamma \rightarrow \Delta$ of clauses to not contain any base-sort constant symbols does not pose any restriction to expressiveness. Every base-sort constant symbol c in the free part can safely be replaced by a fresh base-sort variable x_c when an atomic constraint $x_c = c$ is added to the constraint part of the clause (a process known as purification, cf. [BGW94, KW12]).

In the rest of the paper we omit the phrase “over the BSR fragment with simple bounds” when talking about clauses and clause sets, although we mainly restrict our considerations to this fragment.

A *hierarchic interpretation* over the *hierarchic specification* $\langle \langle \Sigma_{\text{LA}}, \mathfrak{M} \rangle, \langle \{\mathcal{S}, \mathcal{R}\}, \Omega, \Pi \rangle \rangle$ is an algebra \mathcal{A} which *extends* a model $\mathcal{M} \in \mathfrak{M}$, i.e. \mathcal{A} and \mathcal{M} interpret the base sort and all constant and predicate symbols in $\Omega_{\text{LA}} \cup \Omega_{\text{LA}}^\alpha$ and Π_{LA} in exactly the same way. Moreover, \mathcal{A} comprises a nonempty domain $\mathcal{S}^{\mathcal{A}}$, assigns to each constant symbol $c : \mathcal{S}$ in Ω a domain element $c^{\mathcal{A}} \in \mathcal{S}^{\mathcal{A}}$ and interprets every predicate symbol $P : \xi_1 \times \dots \times \xi_m \in \Pi$ by a set $P^{\mathcal{A}} \subseteq \xi_1^{\mathcal{A}} \times \dots \times \xi_m^{\mathcal{A}}$. Summing up, \mathcal{A} extends the standard model of linear arithmetic and adopts the standard approach to semantics of (sorted) first-order logics when interpreting the free part of clauses.

Given a hierarchic interpretation \mathcal{A} , a *variable assignment* is a sort-respecting total mapping $\beta : V_{\mathcal{R}} \cup V_{\mathcal{S}} \rightarrow \mathbb{R} \cup \mathcal{S}^{\mathcal{A}}$ so that $\beta(x) \in \mathbb{R}$ for every variable $x \in V_{\mathcal{R}}$ and $\beta(u) \in \mathcal{S}^{\mathcal{A}}$ for every $u \in V_{\mathcal{S}}$. We write $\mathcal{A}(\beta)(s)$ to mean the *value of the term s under \mathcal{A} with respect to the variable assignment β* . In accordance with the notation used so far, we thus define $\mathcal{A}(\beta)(v) := \beta(v)$ for every variable v and $\mathcal{A}(\beta)(c) := c^{\mathcal{A}}$ for every constant symbol c . As usual, we use the symbol \models to denote truth under a hierarchic interpretation \mathcal{A} , possibly with respect to a variable assignment β . In detail, we have the following for atomic constraints, equational and nonequational free atoms and clauses, respectively:

- $\mathcal{A}, \beta \models s \triangleleft s'$ if and only if $\mathcal{A}(\beta)(s) \triangleleft \mathcal{A}(\beta)(s')$,
- $\mathcal{A}, \beta \models s \approx s'$ if and only if $\mathcal{A}(\beta)(s) = \mathcal{A}(\beta)(s')$,
- $\mathcal{A}, \beta \models P(s_1, \dots, s_m)$ if and only if
 $\langle \mathcal{A}(\beta)(s_1), \dots, \mathcal{A}(\beta)(s_m) \rangle \in P^{\mathcal{A}}$,

- $\mathcal{A}, \beta \models \Lambda \parallel \Gamma \rightarrow \Delta$ if and only if
 - $\mathcal{A}, \beta \not\models s \triangleleft s'$ for some atomic constraint $(s \triangleleft s') \in \Lambda$,
 - $\mathcal{A}, \beta \not\models A$ for some free atom $A \in \Gamma$, or
 - $\mathcal{A}, \beta \models B$ for some free atom $B \in \Delta$.

The variables occurring in clauses shall be universally quantified. Therefore, given a clause C , we call \mathcal{A} a *hierarchic model* of C , denoted $\mathcal{A} \models C$, if and only if $\mathcal{A}, \beta \models C$ holds for every variable assignment β . For clause sets N , we say \mathcal{A} is a *hierarchic model* of N , also denoted $\mathcal{A} \models N$, if and only if \mathcal{A} is a model of all clauses in N . We call a clause C (a clause set N) *satisfiable w.r.t.* \mathfrak{M} if and only if there exists a hierarchic model \mathcal{A} of C (of N). Most of the time, we will omit the explicit reference to \mathfrak{M} , although we shall only consider models as satisfying that extend the standard model of linear real arithmetic.

From now on, we implicitly base all further considerations on a hierarchic specification $\langle \langle \Sigma_{\text{LA}}, \mathfrak{M} \rangle, \langle \{\mathcal{S}, \mathcal{R}\}, \Omega, \Pi \rangle \rangle$ whose extension part $\langle \{\mathcal{S}, \mathcal{R}\}, \Omega, \Pi \rangle$ fulfills the requirements stipulated in Definition 1 and which we simply take for granted. Often the sets $\Omega_{\text{LA}} \setminus \mathbb{R}$, Ω and Π will coincide with the constant and predicate symbols that occur in a given clause set, and we thus assume a proper definition of the hierarchic specification.

Substitutions σ shall be defined in the standard way as sort-respecting mappings from variables to terms over our underlying signature. The *restriction of the domain of a substitution σ to a set V of variables* is denoted by $\sigma|_V$ and shall be defined so that $v\sigma|_V := v\sigma$ for every $v \in V$ and $v\sigma|_V = v$ for every $v \notin V$. While the application of a substitution σ to terms, atoms and multisets thereof can be defined as usual, we need to be more specific for clauses. Consider a clause $C = \Lambda \parallel \Gamma \rightarrow \Delta$ and let x_1, \dots, x_k denote all base-sort variables occurring in C for which $x_i\sigma \neq x_i$. We then set $C\sigma := \Lambda\sigma, x_1 = x_1\sigma, \dots, x_k = x_k\sigma \parallel (\Gamma \rightarrow \Delta)\sigma|_{V_S}$. *Simultaneous substitution* of $k \geq 1$ distinct variables v_1, \dots, v_k with terms s_1, \dots, s_k shall be denoted in vector notation $[v_1, \dots, v_k / s_1, \dots, s_k]$ (or $[\bar{v} / \bar{s}]$ for short), where we require every s_i to be of the same sort as v_i , of course.

Consider a clause C and let σ be an arbitrary substitution. The clause $C\sigma$ and variable-renamed variants thereof are called *instances* of C . A term s is called *ground*, if it does not contain any variables. A clause C shall be called *essentially ground* if it does not contain free-sort variables and for every base-sort variable x occurring in C , there is an atomic constraint $x = d$ in C for some constant symbol $d \in \Omega_{\text{LA}} \cup \Omega_{\text{LA}}^\alpha$. A clause set N is *essentially ground* if all the clauses it contains are essentially ground.

We can restrict the syntactic form of clauses even further without limiting expressiveness.

Definition 2 (Normal Form of Clauses and Clause Sets). A clause $\Lambda \parallel \Gamma \rightarrow \Delta$ is in *normal form* if all base-sort variables which occur in Λ do also occur in $\Gamma \rightarrow \Delta$. A clause set N over the BSR fragment with simple bounds is in *normal form* if all clauses in N are in normal form and pairwise variable disjoint.

Let us briefly clarify why the above requirement on clauses does not limit expressiveness. Any base-sort variable x not fulfilling the stated requirement can be removed from the clause $\Lambda \parallel \Gamma \rightarrow \Delta$ by existential quantifier elimination methods that transform Λ into an equivalent constraint Λ' in which x does not occur.¹ Moreover, Λ' can be constructed in such a way that it contains only atomic constraints of the form admitted in Definition 1 and so that no variables or constant symbols other than the ones in Λ are necessary.

Given a clause set N , we will use the following notation: the set of all constant symbols occurring in N shall be denoted by $\text{consts}(N)$. While the set $\text{bconsts}(N)$ exclusively contains all base-sort constant symbols from Ω_{LA} that occur in N , all base-sort constant symbols α_t appearing in N shall be collected in the set $\alpha\text{consts}(N) := \text{consts}(N) \cap \Omega_{\text{LA}}^\alpha$. The set of all free-sort constant

¹Methods for the elimination of existentially quantified real variables include Fourier-Motzkin variable elimination [DE73], the Loos-Weispfenning procedure [LW93] and many others, see e.g. Chapter 5 in [KS08] for further details.

symbols in N is called $\text{fconsts}(N)$. Altogether, the sets $\text{bconsts}(N)$, $\alpha\text{consts}(N)$ and $\text{fconsts}(N)$ together form a partitioning of $\text{consts}(N)$ for every clause set N . We moreover denote the set of all variables occurring in a clause C (clause set N) by $\text{vars}(C)$ ($\text{vars}(N)$).

3 Instantiation of Base-Sort Variables

We first summarize the overall approach described in this section in an intuitive way. To keep the informal exposition simple, we pretend that all base-sort constant symbols are taken from \mathbb{R} and thus are interpreted by their canonical value in all hierachic interpretations. Consequently, we can speak of real values instead of constant symbols, and even refer to improper values such as $-\infty$ (a “sufficiently small” real value) and $r + \varepsilon$ (a real value “slightly larger than r but not too large”). A formal treatment with proper definitions will follow.

Given a finite clause set N , we aim at partitioning the reals \mathbb{R} into finitely many partitions so that whenever N is satisfiable, there exists a hierachic model \mathcal{A} of N

(\mathfrak{Pr}) whose interpretation of predicate symbols does not distinguish between values within the same partition.

As soon as we found such a finite partitioning \mathcal{P} , we pick one real value $r_p \in p$ as representative from every partition $p \in \mathcal{P}$. The following observation motivates why we can use those representatives instead of using universally quantified variables: given a clause C that contains a base-sort variable x , and given a set $\{c_1, \dots, c_k\}$ of constant symbols such that $\{c_1^{\mathcal{A}}, \dots, c_k^{\mathcal{A}}\} = \{r_p \mid p \in \mathcal{P}\}$ it holds

$$\mathcal{A} \models C \iff \mathcal{A} \models \{C[x/c_i] \mid 1 \leq i \leq k\}. \quad (1)$$

The equivalence claims that we can transform universal quantification over the base domain into finite conjunction over all representatives of partitions in \mathcal{P} . The formal version of this statement is given in Lemma 13, and we will see that hierachic models complying with property (\mathfrak{Pr}) play a key role in its proof. Since \mathcal{P} is supposed to be finite, the resulting set of instances $\{C[x/c_i] \mid 1 \leq i \leq k\}$ is finite, too.

It turns out that the notion of *elimination sets* described by Loos and Weispfenning in [LW93] (in the context of quantifier elimination for linear arithmetic) can be adapted to yield reasonable sets of representatives from which we can construct a finite partitioning exhibiting the described characteristics. In this case the partitions are intervals on the real axis. Intuitively speaking, we start with the partitioning \mathcal{P}_0 consisting of a single partition $\mathcal{P}_0 = \{(-\infty, +\infty)\}$. This initial partition shall be represented by $-\infty$ (the “default representative”). We then successively extract from the given clause set N larger and larger points r on the real axis at which we have to introduce a new boundary, cut off the interval that is currently unbounded from above at that boundary and introduce the cut-off part as a new interval to the partitioning. For instance, an interval $[r', +\infty)$ might be cut into two parts $[r', r]$ and $(r, +\infty)$ for some point $r \geq r'$. Both parts become then new partitions in the partitioning and while the interval $[r', r]$ will be represented by r' , the other partition $(r, +\infty)$ will have $r + \varepsilon$ as its representative. At the end of this process the overall result will be a finite partitioning of the real numbers with the desired properties.

In fact, we will operate on a more fine-grained level than described so far, as we define such partitionings independently for certain groups of base-sort variables. The possible benefit may be a significant decrease in the number of necessary instances. But there is even more potential for savings. The complete line of definitions and arguments will be laid out in detail for one direction of instantiation along the real axis, namely in the positive direction starting from $-\infty$ and going on to larger and larger instantiation points. However, one could as well proceed in the opposite way, starting from $+\infty$ and becoming smaller and smaller thereafter. While theoretically this duality is not worth much more than a side note, it appears to be quite interesting from a practical point of view. As it turns out, one can choose the direction of instantiation independently for each base-sort variable that needs to be instantiated. Hence, one could always pick the direction that results in less instantiation points. Such a strategy might again considerably cut down the number

of instances in the resulting clause set and therefore might lead to shorter processing times when applying automated reasoning procedures.

Formally, the aforementioned representatives are induced by constant symbols when interpreted under a hierachic interpretation. Independently of any interpretation these constant symbols will serve as symbolic *instantiation points*, i.e. they will be used to instantiate base-sort variables similar to the c_i in equivalence (1). We start off by defining the set of instantiation points that need to be considered for every base-sort variable. This set depends on the constraints affecting such a variable. However, we first need to develop a proper notion of what it means for a base-sort variable to be affected by a constraint. The following example shall illustrate the involved issues.

Example 3. Consider the following clauses:

$$\begin{array}{lll} x_1 \neq -5 & \parallel & \rightarrow T(x_1), Q(x_1, x_2) , \\ y_1 < 2, y_2 < 0 & \parallel & \rightarrow Q(y_1, y_2) , \\ z_1 \geq 6 & \parallel T(z_1) & \rightarrow \square . \end{array}$$

Obviously, the variables x_1 , y_1 , y_2 and z_1 are affected by the constraints in which they occur explicitly. In the given clauses variables are just names addressing argument positions of predicate symbols. Thus, it is more suitable to speak of the *argument position* $\langle T, 1 \rangle$ instead of variables x_1 and z_1 that occur as the first argument of predicate symbol T in the first and third clause. Speaking in such terms, argument position $\langle T, 1 \rangle$ is directly affected by the constraints $x_1 \neq -5$ and $z_1 \geq 6$, argument position $\langle Q, 1 \rangle$ is directly affected by $x_1 \neq -5$ and $y_1 < 2$, and finally $\langle Q, 2 \rangle$ is affected by $y_2 < 0$. As soon as we take logical consequences into account, the notion of “being affected” needs to be extended. The above clause set, for instance, logically entails the clause $x \geq 6 \parallel \rightarrow Q(x, y)$. Hence, although not *directly* affected by the constraint $z_1 \geq 6$ in the clause set, the argument position $\langle Q, 1 \rangle$ is still indirectly subject to this constraint. The source of this effect lies in the first clause as it establishes a connection between argument positions $\langle T, 1 \rangle$ and $\langle Q, 1 \rangle$ via the simultaneous occurrence of variable x_1 in both argument positions.

One lesson learned from the example is that argument positions can be connected by variable occurrences. Such links in a clause set N shall be expressed by the relation \leftrightharpoons_N .

Definition 4 (Connections between Argument Positions). Let N be a clause set in normal form. We define the relation \leftrightharpoons_N to be the smallest equivalence relation over pairs in $\Pi \times \mathbb{N}$ such that $\langle Q, j \rangle \leftrightharpoons_N \langle P, i \rangle$ whenever there is a clause in N containing free atoms $Q(\dots, v, \dots)$ and $P(\dots, v, \dots)$ in which the variable v occurs at the j -th and i -th argument position, respectively. (Note that $Q = P$ or $j = i$ is possible.)

The relation \leftrightharpoons_N induces the set $\{[\langle P, i \rangle]_{\leftrightharpoons_N} \mid P/m \in \Pi, 1 \leq i \leq m\}$ of equivalence classes. To simplify notation a bit, we write $[\langle P, i \rangle]$ instead of $[\langle P, i \rangle]_{\leftrightharpoons_N}$ when the set N is clear from the context.

As we have argued earlier, it is more precise to speak of argument positions rather than variables, since their names are of no particular relevance. Nevertheless, variable names are a syntactical necessity. The following definition is supposed to provide a means to address the argument position class a variable stands for.

Definition 5 (Argument Position Class). Let N be a clause set in normal form. Consider a variable v that occurs at the i -th argument position of a free atom $P(\dots, v, \dots)$ in N . We denote the *equivalence class of argument positions* v is related to in N by $ap_N(v)$, i.e. $ap_N(v) := [\langle P, i \rangle]_{\leftrightharpoons_N}$. If v is a free-sort variable that exclusively occurs in equations $v \approx s$ in N , we set $ap_N(v) := \emptyset$.

There is a crucial subtlety in this definition that guarantees well-definedness, namely that the clauses in N are variable disjoint. Given a clause $C \in N$ and a variable v which occurs in different argument positions $\langle Q, j \rangle$ and $\langle P, i \rangle$ in C , the definition of \leftrightharpoons_N entails $\langle Q, j \rangle \leftrightharpoons_N \langle P, i \rangle$. Therefore, $[\langle Q, j \rangle]$ and $[\langle P, i \rangle]$ are identical. Since v does not occur in any other clause in N , the class $ap_N(v)$ is well-defined to be $[\langle P, i \rangle]$.

Next, we collect the instantiation points that are necessary to eliminate base-sort variables by means of finite instantiation. In order to do this economically, we rely on the same idea that also keeps elimination sets in [LW93] comparatively small.

Definition 6 (Instantiation Points for Base-Sort Argument Positions). Let N be a clause set in normal form and let $P : \xi_1 \times \dots \times \xi_m \in \Pi$ be a free predicate symbol occurring in N . Let $J := \{i \mid \xi_i = \mathcal{R}, 1 \leq i \leq m\}$ be the indices of P 's base-sort arguments. For every $i \in J$, we define $\mathcal{I}_{P,i,N}$ to be the smallest set fulfilling

- (i) $d \in \mathcal{I}_{P,i,N}$ if there exists a clause C in N containing an atom $P(\dots, x, \dots)$ in which x occurs as the i -th argument and a constraint $x = d$ or $x \geq d$ with $d \in \Omega_{LA}$ appears in C , and
- (ii) $\alpha_{d+\varepsilon} \in \mathcal{I}_{P,i,N}$ if there exists a clause C in N containing an atom of the form $P(\dots, x, \dots)$, in which x is the i -th argument and a constraint of the form $x \neq d$ or $x > d$ or $x = \alpha_{d+\varepsilon}$ with $d \in \Omega_{LA}$ appears in C .

The most apparent peculiarity about this definition is that atomic constraints of the form $x < d$ and $x \leq d$ are completely ignored when collecting instantiation points for x 's argument position. First of all, this is one of the aspects that makes this definition interesting from the efficiency point of view, because the number of instances that we have to consider might decrease considerably in this way. To develop an intuitive understanding why it is enough to consider constraints $x \triangleleft d$ with $\triangleleft \in \{=, \neq, \geq, >\}$ when collecting instantiation points, the following example may help.

Example 7. Consider two clauses $C = x > 2, x \leq 5 \parallel \rightarrow T(x)$ and $D = x < 0 \parallel T(x) \rightarrow \square$. Recall that we are looking for a finite partitioning \mathcal{P} of \mathbb{R} so that we can construct a hierachic model \mathcal{A} of the clause set $\{C, D\}$ that complies with (\mathfrak{Pr}) , i.e. for every partition $p \in \mathcal{P}$ and arbitrary real values $r_1, r_2 \in p$ it shall hold $r_1 \in T^{\mathcal{A}}$ if and only if $r_2 \in T^{\mathcal{A}}$. Of course, there exist infinitely many candidates for \mathcal{P} . A natural one is $\{(-\infty, 0), [0, 2], (2, 5], (5, +\infty)\}$ which takes every atomic constraint in C and D into account. Correspondingly, we find a candidate predicate for $T^{\mathcal{A}}$, namely the interval $(2, 5]$, so that \mathcal{A} is a hierachic model of C and D alike and it obeys (\mathfrak{Pr}) with respect to the proposed partitioning \mathcal{P} .

But there are other interesting possibilities, too, for instance, the more coarse-grained partitioning $\{(-\infty, 2], (2, +\infty)\}$ together with the predicate $T^{\mathcal{A}} = (2, +\infty)$. This latter candidate partitioning completely ignores the constraints $x < 0$ and $x \leq 5$ that constitute upper bounds on x . Dually, we could have concentrated on the upper bounds instead (completely ignoring the lower bounds). This would have lead to the partitioning $\{(-\infty, 0), [0, 5], (5, +\infty)\}$ and candidate predicates $T^{\mathcal{A}} = [0, 5]$ (or $T^{\mathcal{A}} = [0, +\infty)$). Both ways are possible, but the first one induces a simpler partitioning.

The example has revealed quite some freedom in choosing an appropriate partitioning of the reals. A larger number of partitions directly leads to a larger number of representatives (one for each interval). In fact, we use the collected instantiation points as representatives and taken together they induce the partitioning (as we will see in the next definition). It is due to this direct correspondence of partitions and instantiation points that a more fine-grained partitioning entails a larger number of instances that need to be considered. Hence, regarding efficiency, it is of high interest to keep the partitioning coarse.

Definition 8 (Instantiation Points for Argument Position Classes and Induced Partitioning of \mathbb{R}). Let N be a clause set in normal form and let \mathcal{A} be a hierachic interpretation (over the same signature). For every equivalence class $[\langle P, i \rangle]$ induced by \equiv_N we define the following:

The set $\mathcal{I}_{[\langle P, i \rangle], N}$ of *instantiation points for* $[\langle P, i \rangle]$ is defined by

$$\mathcal{I}_{[\langle P, i \rangle], N} := \{\alpha_{-\infty}\} \cup \bigcup_{\langle Q, j \rangle \in [\langle P, i \rangle]} \mathcal{I}_{Q,j,N} .$$

The sequence r_1, \dots, r_k shall comprise all real values in the set $\{c^{\mathcal{A}} \mid c \in \mathcal{I}_{[\langle P, i \rangle], N} \cap \Omega_{LA}$ or $\alpha_{c+\varepsilon} \in \mathcal{I}_{[\langle P, i \rangle], N}\}$ ordered so that $r_1 < \dots < r_k$.

Given a real number r , we say $\mathcal{I}_{[\langle P, i \rangle], N}$ \mathcal{A} -covers r if there exists an instantiation point $c \in \mathcal{I}_{[\langle P, i \rangle], N} \cap \Omega_{\text{LA}}$ with $c^{\mathcal{A}} = r$; analogously $(r + \varepsilon)$ is \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$ if there is an instantiation point $\alpha_{c+\varepsilon} \in \mathcal{I}_{[\langle P, i \rangle], N}$ with $c^{\mathcal{A}} = r$.

The partitioning $\mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$ of the reals into finitely many intervals shall be the smallest partitioning (smallest w.r.t. the number of partitions) fulfilling the following requirements:

- (i) If r_1 is \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$, then $(-\infty, r_1) \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$; otherwise, $(-\infty, r_1] \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$.
- (ii) For every j , $1 \leq j \leq k$, if r_j and $r_j + \varepsilon$ are both \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$, then $[r_j, r_j] = \{r_j\} \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$.
- (iii) For every ℓ , $1 \leq \ell < k$,
 - (iii.i) if $r_\ell + \varepsilon$ and $r_{\ell+1}$ are both \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$, then $(r_\ell, r_{\ell+1}) \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$,
 - (iii.ii) if $r_\ell + \varepsilon$ is \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$ but $r_{\ell+1}$ is not, then $(r_\ell, r_{\ell+1}] \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$,
 - (iii.iii) if $r_\ell + \varepsilon$ is not \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$ but $r_{\ell+1}$ is, then $[r_\ell, r_{\ell+1}) \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$,
 - (iii.iv) if neither $r_\ell + \varepsilon$ nor $r_{\ell+1}$ is \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$, then $[r_\ell, r_{\ell+1}] \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$.
- (iv) If $r_k + \varepsilon$ is \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$, then $(r_k, +\infty) \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$; otherwise $[r_k, +\infty) \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$.

Please note that partitionings as described in the definition do always exist, and do not contain empty partitions. Moreover, it is worth to notice that for all instantiation points c and $\alpha_{c+\varepsilon}$ in a set $\mathcal{I}_{Q, j, N}$ we have $c \in \Omega_{\text{LA}}$. Hence, the concrete values assigned to constant symbols α_t do not contribute to the respective partitionings of \mathbb{R} .

We now fix the semantics of constant symbols α_t by giving appropriate axioms that make precise what it means for $\alpha_{-\infty}$ to be “small enough” and for $\alpha_{c+\varepsilon}$ to be “a little larger than c but not too large” under any hierachic model \mathcal{A} that satisfies the corresponding axioms.

Definition 9 (Axioms for Instantiation Points α_t). Let \mathcal{I} be a set of instantiation points and $\mathcal{C} \subseteq \Omega_{\text{LA}}$ be a set of base-sort constant symbols. We define the *set of axioms*

$$\begin{aligned} \text{Ax}_{\mathcal{I}, \mathcal{C}} := & \{ \alpha_{-\infty} < c \mid \alpha_{-\infty} \in \mathcal{I} \text{ and } c \in \mathcal{C} \} \cup \{ d < \alpha_{d+\varepsilon} \mid \alpha_{d+\varepsilon} \in \mathcal{I} \} \\ & \cup \{ d < c \rightarrow \alpha_{d+\varepsilon} < c \mid \alpha_{d+\varepsilon} \in \mathcal{I} \text{ and } c \in \mathcal{C} \setminus \{d\} \} \\ & \cup \{ d = c \rightarrow \alpha_{d+\varepsilon} = \alpha_{c+\varepsilon} \mid \alpha_{d+\varepsilon} \in \mathcal{I} \text{ and } c \in \mathcal{C} \setminus \{d\} \}. \end{aligned}$$

The axioms we just introduced are clearly not in the admissible clause form. But they can easily be transformed into proper clauses with empty free parts. For convenience, we stick to the notation given here, but keep in mind that formally the axioms have a form in accordance with Definition 1.

The following proposition shows that every hierachic interpretation can be turned into a model of a given set of axioms of the above shape by modifying the values assigned to constant symbols α_t . The proposition relies on three facts concerning the ordering $(\mathbb{R}, <)$: totality of $<$, the lack of minimal and maximal elements, and the density of $(\mathbb{R}, <)$, i.e. for arbitrary reals $r_1 < r_2$ there is an r' between them.

Proposition 10. Let \mathcal{I} be a nonempty set of instantiation points and let \mathcal{C} be a nonempty set of base-sort constant symbols from Ω_{LA} . Given an arbitrary hierachic interpretation \mathcal{A} , we can construct a hierachic model \mathcal{B} so that (ii) $\mathcal{B} \models \text{Ax}_{\mathcal{I}, \mathcal{C}}$ and \mathcal{B} and \mathcal{A} differ only in the interpretation of the constant symbols in $\text{acons}(\text{Ax}_{\mathcal{I}, \mathcal{C}})$.

Proof. The model \mathcal{B} can be derived from \mathcal{A} by redefining the interpretation of all constant symbols $\alpha_t \in \mathcal{I}$ as follows:

$$\begin{aligned} \alpha_{-\infty}^{\mathcal{B}} &:= \min\{c^{\mathcal{A}} \mid c \in \mathcal{C}\} - 1, \\ \alpha_{d+\varepsilon}^{\mathcal{B}} &:= \frac{1}{2}(d^{\mathcal{A}} + \min(\{c^{\mathcal{A}} \mid c \in \mathcal{C}, c^{\mathcal{A}} > d^{\mathcal{A}}\} \cup \{d^{\mathcal{A}} + 1\})). \end{aligned}$$

Everything else is taken over from \mathcal{A} . □

Previously, we have been speaking of representatives of partitions p in a partitioning \mathcal{P} of the reals. As far as we know by now, these are just real values $r_p \in p$ from the specific interval they represent. The next lemma shows that we can find constant symbols c_p in the set of instantiation points to address them. Furthermore, we will see where the values of these constant symbols lie within the respective interval.

In order to have a compact notation at hand when doing case distinctions on intervals, we use “(·” to stand for “(” as well as for “[”. Analogously, “)·” is used to address the two cases “)” and “]” at the same time.

Lemma 11. Let N be a clause set in normal form and let \mathcal{A} be a hierachic model of $\text{Ax}_{\mathcal{I}_{[\langle P, i \rangle], N}, \text{bconsts}(N)}$ for an arbitrary argument position pair $\langle P, i \rangle$. For every partition $p \in \mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$ we find some constant symbol $c_{[\langle P, i \rangle], p} \in \mathcal{I}_{[\langle P, i \rangle], N}$ so that $c_{[\langle P, i \rangle], p}^{\mathcal{A}} \in p$ and

- (i) if $p = (-\infty, r_u)$ or $p = (-\infty, +\infty)$, then $c_{[\langle P, i \rangle], p}^{\mathcal{A}} < d^{\mathcal{A}}$ for every $d \in \text{bconsts}(N)$,
- (ii) if $p = [r_\ell, r_u)$ or $p = [r_\ell, +\infty)$, then $c_{[\langle P, i \rangle], p}^{\mathcal{A}} = r_\ell$.
- (iii) if $p = (r_\ell, r_u)$ or $p = (r_\ell, +\infty)$, then $r_\ell < c_{[\langle P, i \rangle], p}^{\mathcal{A}}$ and $c_{[\langle P, i \rangle], p}^{\mathcal{A}} < d^{\mathcal{A}}$ for every $d \in \text{bconsts}(N)$ with $r_\ell < d^{\mathcal{A}}$.

Proof. We proceed by case distinction on the form of the partition p .

Case $p = (-\infty, r_u)$ for some real value r_u . By construction of $\mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$, there is a constant symbol $e \in \text{bconsts}(N)$ so that $e^{\mathcal{A}} = r_u$. Moreover, we find the instantiation point $\alpha_{-\infty}$ in $\mathcal{I}_{[\langle P, i \rangle], N}$ and thus $\text{Ax}_{\mathcal{I}_{[\langle P, i \rangle], N}, \text{bconsts}(N)}$ contains the axiom $\alpha_{-\infty} < d$ for every constant symbol $d \in \text{bconsts}(N)$, in particular $\alpha_{-\infty} < e$. Hence, $\alpha_{-\infty}^{\mathcal{A}} \in (-\infty, r_u)$.

Case $p = (-\infty, +\infty)$. We find the instantiation point $\alpha_{-\infty}$ in $\mathcal{I}_{[\langle P, i \rangle], N}$, and obviously $\alpha_{-\infty}^{\mathcal{A}}$ lies in p . Moreover, $\text{Ax}_{\mathcal{I}_{[\langle P, i \rangle], N}, \text{bconsts}(N)}$ contains the axiom $\alpha_{-\infty} < d$ for every constant symbol $d \in \text{bconsts}(N)$.

Case $p = [r_\ell, r_u)$ for real values r_ℓ, r_u with $r_\ell \leq r_u$.

If p is a point interval $[r_\ell, r_\ell]$, then we find a constant symbol $e \in \mathcal{I}_{[\langle P, i \rangle], N} \cap \Omega_{\text{LA}}$ so that $p = \{e^{\mathcal{A}}\}$.

If p is not a point interval, i.e. $r_\ell < r_u$, then the definition of $\mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$ entails the existence of a constant symbol $e \in \text{bconsts}(N)$ such that $e^{\mathcal{A}} = r_\ell$ and either $e \in \mathcal{I}_{[\langle P, i \rangle], N}$ or $\alpha_{e+\varepsilon} \in \mathcal{I}_{[\langle P, i \rangle], N}$. But since the partition p is not a point interval, $r_\ell + \varepsilon$ cannot be \mathcal{A} -covered by $\mathcal{I}_{[\langle P, i \rangle], N}$, and thus $\alpha_{e+\varepsilon}$ cannot be in $\mathcal{I}_{[\langle P, i \rangle], N}$. Consequently, e must be in $\mathcal{I}_{[\langle P, i \rangle], N}$ and we can choose $c_{[\langle P, i \rangle], p} := e$.

The case of $p = [r_\ell, +\infty)$ can be argued analogously to the previous case.

Case $p = (r_\ell, r_u)$ for real values r_ℓ, r_u with $r_\ell < r_u$.

By construction of $\mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$, we find some instantiation point $\alpha_{e+\varepsilon} \in \mathcal{I}_{[\langle P, i \rangle], N}$ so that $e^{\mathcal{A}} = r_\ell$. Moreover, there exists a second constant symbol $e' \in \text{bconsts}(N)$ for which $e'^{\mathcal{A}} = r_u$. As a consequence of $\alpha_{e+\varepsilon} \in \mathcal{I}_{[\langle P, i \rangle], N}$ the set $\text{Ax}_{\mathcal{I}_{[\langle P, i \rangle], N}, \text{bconsts}(N)}$ contains the axioms $e < \alpha_{e+\varepsilon}$ and $e < d \rightarrow \alpha_{e+\varepsilon} < d$ for every constant symbol $d \in \text{bconsts}(N)$, in particular for $d = e'$. And since \mathcal{A} is a model of $\text{Ax}_{\mathcal{I}_{[\langle P, i \rangle], N}, \text{bconsts}(N)}$, requirement (iii) is satisfied and furthermore $\alpha_{e+\varepsilon}^{\mathcal{A}} \in p$ follows, since $e^{\mathcal{A}} < \alpha_{e+\varepsilon}^{\mathcal{A}} < e'^{\mathcal{A}}$.

Case $p = (r_\ell, +\infty)$. There must be an instantiation point $\alpha_{d+\varepsilon}$ in $\mathcal{I}_{[\langle P, i \rangle], N}$ such that $e^{\mathcal{A}} = r_\ell$. As we have already done in Definition 8, we denote by r_1, \dots, r_k all real values in ascending order which \mathcal{A} assigns to constant symbols $d \in \text{bconsts}(N)$ that either themselves are instantiation points in $\mathcal{I}_{[\langle P, i \rangle], N}$ or for which $\alpha_{d+\varepsilon}$ occurs in $\mathcal{I}_{[\langle P, i \rangle], N}$. By construction of $\mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$, we know $r_\ell = r_k > r_{k-1} > \dots > r_1$, and thus there is no base-sort constant symbol e' in

$\text{bconsts}(N)$ fulfilling $r_\ell < e'^{\mathcal{A}}$. Consequently, (iii) is satisfied if we choose $c_{[(P,i)],p} := \alpha_{e+\varepsilon}$. Finally, we may conclude $\alpha_{e+\varepsilon}^{\mathcal{A}} \in p$, because $\text{Ax}_{\mathcal{I}_{[(P,i)]}, N, \text{bconsts}(N)}$ contains the axiom $e < \alpha_{e+\varepsilon}$.

□

We have arrived at one of the core results of the present paper. The next lemma shows that we can eliminate base-sort variables x from clauses C in a clause set N by replacing C with finitely many instances in which x is substituted with the instantiation points that we computed for x . The resulting clause set shall be called \widehat{N}_x . In addition, the axioms that stipulate the meaning of newly introduced constant symbols α_t need to be added to \widehat{N}_x . Iterating this step for every base-sort variable in N eventually leads to a clause set that is essentially ground with respect to the constraint parts of the the clauses it contains (free-sort variables need to be treated separately, of course, see Section 4).

The line of argument leading from a hierachic model of the original clause set N to a hierachic model of the modified clause set \widehat{N}_x (with instances and additional axioms) is almost trivial. The converse direction, however, rests on a model construction that yields a hierachic model \mathcal{B} which complies with property (\mathfrak{Pr}) discussed in the beginning of the current section. Given a hierachic model \mathcal{A} of the instantiated clause set \widehat{N}_x , we construct the partitioning $\mathcal{P}_{\text{ap}_N(x), N, \mathcal{A}}$ based on the argument position class associated to x in the original clause set N . By virtue of Lemma 11, we know that each partition p in the partitioning is represented by an instantiation point c_p in $\mathcal{I}_{\text{ap}_N(x), N}$ and for each of these instantiation points there is one clause in the modified clause set \widehat{N}_x in which x is instantiated by c_p . Since \mathcal{B} is supposed to comply with (\mathfrak{Pr}) , i.e. it shall not distinguish between real values that stem from the same partition, the information how the model \mathcal{A} treats the representative of a partition can be transferred to all values from this partition. An example might be best suited to illustrate the key ideas.

Example 12. Consider the clause set $N = \{x > 2, z = 4 \parallel Q(x, z) \rightarrow T(x)\}$. With respect to N there are two instantiation points for variable x , namely $\alpha_{-\infty}$ and $\alpha_{2+\varepsilon}$. This will lead to the set of instances and axioms

$$\begin{aligned}\widehat{N}_x = & \{\alpha_{-\infty} > 2, z = 4, x = \alpha_{-\infty} \parallel Q(x, z) \rightarrow T(x), \\ & \alpha_{2+\varepsilon} > 2, z = 4, x = \alpha_{2+\varepsilon} \parallel Q(x, z) \rightarrow T(x)\} \\ & \cup \text{Ax}_{\{\alpha_{-\infty}, \alpha_{2+\varepsilon}\}, \{2, 4\}}\end{aligned}$$

where the axioms express the fact that the value of $\alpha_{-\infty}$ is strictly smaller than 2 and the value of $\alpha_{2+\varepsilon}$ shall lie within the interval $(2, 4)$ in any hierachic model of \widehat{N}_x . This already reveals redundancy of the first instance in the presence of the axioms, since the atomic constraint $\alpha_{-\infty} > 2$ is false under every hierachic model of the axiom $\alpha_{-\infty} < 2$ that is contained in $\text{Ax}_{\{\alpha_{-\infty}, \alpha_{2+\varepsilon}\}, \{2, 4\}}$.

We assume to have a hierachic model \mathcal{A} of \widehat{N}_x at hand with $\alpha_{-\infty}^{\mathcal{A}} = 0$ and $\alpha_{2+\varepsilon}^{\mathcal{A}} = 3$. Then the two intervals $(-\infty, 2]$ and $(2, +\infty)$ constitute the partitioning $\mathcal{P}_{\text{ap}_N(x), N, \mathcal{A}}$, and the partitioning $\mathcal{P}_{\text{ap}_N(z), N, \mathcal{A}}$ is given by $\{(-\infty, 4), [4, +\infty)\}$. Moreover, assume that the set $Q^{\mathcal{A}}$ contains the pairs $\langle 0, 0 \rangle$, $\langle 3, 4 \rangle$ and $\langle 7, 4 \rangle$ while the set $T^{\mathcal{A}}$ shall be the union of intervals $(0, 5] \cup [6, 7)$. Clearly, \mathcal{A} is a hierachic model of \widehat{N}_x . (However, \mathcal{A} is not a hierachic model of N , since the pair $\langle 7, 4 \rangle$ is in $Q^{\mathcal{A}}$ but 7 does not belong to $T^{\mathcal{A}}$.)

We now construct a hierachic model \mathcal{B} from \mathcal{A} so that \mathcal{B} complies with (\mathfrak{Pr}) . First of all, \mathcal{B} takes over all values assigned to constant symbols, i.e. $\alpha_{-\infty}^{\mathcal{B}} := \alpha_{-\infty}^{\mathcal{A}}$ and $\alpha_{2+\varepsilon}^{\mathcal{B}} := \alpha_{2+\varepsilon}^{\mathcal{A}}$. The definition of T under \mathcal{B} will be piecewise with respect to the partitioning $\mathcal{P}_{[(T, 1)], N, \mathcal{A}} = \mathcal{P}_{\text{ap}_N(x), N, \mathcal{A}} = \{(-\infty, 2], (2, +\infty)\}$. To do so, we use the idea of representatives: the interval $(-\infty, 2]$ is represented by $\alpha_{-\infty}^{\mathcal{A}} = 0$ and the interval $(2, +\infty)$ has $\alpha_{2+\varepsilon}^{\mathcal{A}} = 3$ as its representative. At the same time, $\alpha_{-\infty}^{\mathcal{A}}$ also represents the interval $(-\infty, 4)$ in partitioning $\mathcal{P}_{[(Q, 2)], N, \mathcal{A}}$, and $[4, +\infty)$ from the same partitioning is represented by the constant 4. Putting the abstract idea of property (\mathfrak{Pr}) into action, the set $T^{\mathcal{B}}$ will be defined so that $(-\infty, 2] \subseteq T^{\mathcal{B}}$ if and only if $\alpha_{-\infty}^{\mathcal{A}} \in T^{\mathcal{A}}$, and $(2, +\infty) \subseteq T^{\mathcal{B}}$ if and only if $\alpha_{2+\varepsilon}^{\mathcal{A}} \in T^{\mathcal{A}}$. For the set $Q^{\mathcal{B}}$ matters are technically more involved but

follow the same scheme. For instance, all pairs $\langle r_1, r_2 \rangle$ with $r_1 \in (2, +\infty)$ and $r_2 \in [4, +\infty)$ shall be in Q^B if and only if the pair $\langle \alpha_{2+\varepsilon}^A, 4 \rangle$ is in Q^A . Consequently, we end up with $T^B = (2, +\infty]$ and $Q^B = \{\langle r_1, r_2 \rangle \mid r_1 \in (-\infty, 2] \text{ and } r_2 \in (-\infty, 4)\} \cup \{\langle r_1, r_2 \rangle \mid r_1 \in (2, +\infty) \text{ and } r_2 \in [4, +\infty)\}$.

The net results is a hierachic interpretation \mathcal{B} that is a model of \widehat{N}_x just as \mathcal{A} is. But beyond that \mathcal{B} is also a hierachic model of the original clause set N . This is by no means a coincidence as the proof of Lemma 13 shows.

Lemma 13. Let N be a clause set in normal form, and assume that N contains all the axioms in $\text{Ax}_{\alpha\text{consts}(N), \text{bconsts}(N)}$ and for every $\alpha_{e+\varepsilon} \in \alpha\text{consts}(N)$ we have $e \in \text{bconsts}(N)$. Suppose there is a clause C in N which contains a base-sort variable x . Let the clause set \widehat{N}_x be constructed as follows: $\widehat{N}_x := (N \setminus \{C\}) \cup \{C[x/c] \mid c \in \mathcal{I}_{\text{ap}_N(x), N}\} \cup \text{Ax}_{\mathcal{I}_{\text{ap}_N(x)}, N, \text{bconsts}(N)}$. The original clause set N is satisfiable if and only if \widehat{N}_x is satisfiable.

Proof. The “only if”-part is almost trivial to show: Let \mathcal{A} be a hierachic model of N . Every axiom in $\text{Ax}_{\mathcal{I}_{\text{ap}_N(x)}, N, \text{bconsts}(N)}$ which does not appear in N concerns constant symbols α_t that do not occur in N . By virtue of Proposition 10, we can derive a hierachic model \mathcal{B} from \mathcal{A} that differs from \mathcal{A} only in how these α_t are interpreted and that is a hierachic model of $\text{Ax}_{\mathcal{I}_{\text{ap}_N(x)}, N, \text{bconsts}(N)} \setminus N$. Thus, $\mathcal{B} \models N$ follows from $\mathcal{A} \models N$. Since the variable x is universally quantified in C , $\mathcal{B} \models C$ entails $\mathcal{B} \models C[x/c]$ for every instance $C[x/c]$. Altogether, we obtain $\mathcal{B} \models \widehat{N}_x$.

The “if”-part requires a more sophisticated argument: Let \mathcal{A} be a hierachic model of \widehat{N}_x . We use \mathcal{A} to construct the hierachic model \mathcal{B} that extends a model in \mathfrak{M} as follows. For the domain \mathcal{S}^B we reuse \mathcal{A} ’s free domain \mathcal{S}^A . For all base-sort and free-sort constant symbols $c \in \text{consts}(N)$, we set $c^B := c^A$. For every predicate symbol $P : \xi_1 \times \dots \times \xi_m \in \Pi$ that occurs in N and for every argument position i , $1 \leq i \leq m$, Lemma 11 guarantees the existence of a base-sort constant symbol $c_{[(P,i)],p} \in \mathcal{I}_{\text{ap}_N(x), N}$ for every partition $p \in \mathcal{P}_{[(P,i)], N, \mathcal{A}}$, such that $c_{[(P,i)],p}^A \in p$ and the requirements (i) to (iii) of the lemma are met. Based on this, we define the family of functions $\varphi_{[(P,i)]} : \mathbb{R} \cup \mathcal{S}^B \rightarrow \mathbb{R} \cup \mathcal{S}^A$ by

$$\varphi_{[(P,i)]}(a) := \begin{cases} c_{[(P,i)],p}^A & \text{if } \xi_i = \mathcal{R} \text{ and } p \in \mathcal{P}_{[(P,i)], N, \mathcal{A}} \\ a & \text{is the partition } a \text{ lies in,} \\ a & \text{if } \xi_i = \mathcal{S}. \end{cases}$$

Using the functions $\varphi_{[(P,i)]}$, we define the P^B so that for all domain elements a_1, \dots, a_m of appropriate sorts $\langle a_1, \dots, a_m \rangle \in P^B$ if and only if $\langle \varphi_{[(P,1)]}(a_1), \dots, \varphi_{[(P,m)]}(a_m) \rangle \in P^A$.

We next show $\mathcal{B} \models N$. Consider any clause $C' = \Lambda' \parallel \Gamma' \rightarrow \Delta'$ in N and let $\beta : V_R \cup V_S \rightarrow \mathbb{R} \cup \mathcal{S}^B$ be an arbitrary variable assignment. From β we derive a special variable assignment β_φ for which we will infer $\mathcal{A}, \beta_\varphi \models C'$ as an intermediate step:

$$\beta_\varphi(v) := \begin{cases} c_{\text{ap}_N(v),p}^A & \text{if } v \in V_R \text{ and } \beta(v) \in p \in \mathcal{P}_{\text{ap}_N(v), N, \mathcal{A}}, \\ \beta(v) & \text{if } v \in V_S, \end{cases}$$

for every variable v . If $C' \neq C$, then \widehat{N}_x already contains C' , and thus $\mathcal{A}, \beta_\varphi \models C'$ must hold. In case of $C' = C$, let p_* be the partition in $\mathcal{P}_{\text{ap}_N(x), N, \mathcal{A}}$ containing the value $\beta(x)$, and let c_* be an abbreviation for $c_{\text{ap}_N(x),p_*}^A$. Due to $\beta_\varphi(x) = c_*^A$ and since \mathcal{A} is a model of the clause $C[x/c_*]$ in \widehat{N}_x , we conclude $\mathcal{A}, \beta_\varphi \models C$. Hence, in any case we can deduce $\mathcal{A}, \beta_\varphi \models C'$. By case distinction on why $\mathcal{A}, \beta_\varphi \models C'$ holds, we may transfer this result to obtain $\mathcal{B}, \beta \models C'$, too.

Case $\mathcal{A}, \beta_\varphi \not\models c \triangleleft d$ for some atomic constraint $c \triangleleft d$ in Λ' with base-sort constant symbols $c, d \in \Omega_{\text{LA}} \cup \Omega_{\text{LA}}^\alpha$ and $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$. Since \mathcal{B} and \mathcal{A} interpret constant symbols in the same way and independently of a variable assignment, we immediately get $\mathcal{B}, \beta \not\models c \triangleleft d$.

Case $\mathcal{A}, \beta_\varphi \not\models (y \triangleleft d) \in \Lambda'$ for an arbitrary base-sort variable y and a constant symbol $d \in \Omega_{\text{LA}}$. This translates to $\beta_\varphi(y) \not\triangleleft d^A$. Let $p \in \mathcal{P}_{\text{ap}_N(y), N, \mathcal{A}}$ be the partition which contains $\beta(y)$ and therefore also $\beta_\varphi(y)$.

If d^A lies outside of p , then $\beta_\varphi(y) \triangleleft d^A$ if and only if $\beta(y) \triangleleft d^A$, since $\beta_\varphi(y) \in p$. Thus, $d^B = d^A$ entails $\mathcal{B}, \beta \not\models y \triangleleft d$.

If p is the point interval $p = \{d^A\}$, then $\beta(y) = \beta_\varphi(y)$, and thus $\mathcal{B}, \beta \not\models y \triangleleft d$.

If $p = (r_\ell, r_u)$ and $r_\ell < d^A \leq r_u$, then $\triangleleft \notin \{<, \leq, \neq\}$, since $\beta_\varphi(y) = c_{\text{ap}_N(y), p}^A < d^A$ (by (ii) and (iii) of Lemma 11). Moreover, we conclude $d \notin \mathcal{I}_{\text{ap}_N(y), N}$, since otherwise p would be of the form $p = [d^A, r_u)$ by the construction of $\mathcal{P}_{\text{ap}_N(y), N, A}$ (requirements (ii), (iii.iii) and (iii.iv)). Therefore, $\triangleleft \notin \{=, \geq\}$, since otherwise the instantiation point d would be in $\mathcal{I}_{\text{ap}_N(y), N}$. This only leaves $\triangleleft = >$. Hence, the constraint $y > d$ occurs in N and thus we find the instantiation point $\alpha_{d+\varepsilon}$ in $\mathcal{I}_{\text{ap}_N(y), N}$. Consequently, requirements (iii.ii) and (iii.iv) of the construction of $\mathcal{P}_{\text{ap}_N(y), N, A}$ entail $p = (r_\ell, d^A]$, which leads to $\beta(y) \leq d^A$ because of $\beta(y) \in p$. Therefore, $\mathcal{B}, \beta \not\models y > d$ must be true.

The cases $p = (r_\ell, +\infty)$, $p = (-\infty, r_u)$ and $(-\infty, +\infty)$ with $r_\ell < d^A \leq r_u$ can be handled by similar arguments.

If $p = [d^A, r_u)$ and $d^A < r_u$, then $\beta_\varphi(y) = c_{\text{ap}_N(y), p}^A = d^A$ by (ii) of Lemma 11. Consequently, $\triangleleft \notin \{\leq, =, \geq\}$. We conclude $\alpha_{d+\varepsilon} \notin \mathcal{I}_{\text{ap}_N(y), N}$, because otherwise $[d^A, r_u)$ would be a point interval, contradicting $d^A < r_u$. Hence, the only remaining possibility is $\triangleleft = <$. But by $\beta(y) \in p$ we deduce $\beta(y) \geq d^A$. Therefore, we clearly get $\mathcal{B}, \beta \not\models y < d$. The case $p = [d^A, +\infty)$ is covered by analogous arguments.

Case $\mathcal{A}, \beta_\varphi \not\models (y = \alpha_t) \in \Lambda'$ for an arbitrary variable y and a constant symbol $\alpha_t \in \Omega_{\text{LA}}^\alpha$. Let $p \in \mathcal{P}_{\text{ap}_N(y), N, A}$ be the partition which contains $\beta(y)$. We immediately conclude $\alpha_t^A \neq c_{\text{ap}_N(y), p}^A$ since $c_{\text{ap}_N(y), p}^A = \beta_\varphi(y) \neq \alpha_t^A$.

If α_t^A does not lie in p , then $\beta(y) \neq \alpha_t^A$, and thus $\mathcal{B}, \beta \not\models y = \alpha_t$.

Assume α_t^A lies in p . By $c_{\text{ap}_N(y), p}^A \in p$ and the facts $\alpha_t^A \neq c_{\text{ap}_N(y), p}^A$ and $\alpha_t^A \in p$, p cannot be a point interval. Moreover, p cannot be of the form $(-\infty, r_u)$, since then $c_{\text{ap}_N(y), p}^A = \alpha_{-\infty}$ and $r_u < \alpha_{c+\varepsilon}^A$ for all $\alpha_{c+\varepsilon} \in \text{acons}(N)$ would follow by construction of $\mathcal{P}_{\text{ap}_N(y), N, A}$ (since $A \models \text{Ax}_{\text{acons}(N), \text{bconsts}(N)}$), contradicting either $\alpha_t \neq c_{\text{ap}_N(y), p}^A$ or $\alpha_t^A \in p$. Consequently, there exists a constant symbol $d \in \text{bconsts}(N)$ so that $\alpha_t = \alpha_{d+\varepsilon}$. Hence, the atomic constraint $y = \alpha_{d+\varepsilon}$ contributes the instantiation point $\alpha_{d+\varepsilon}$ to $\mathcal{I}_{\text{ap}_N(y), N}$.

The constant symbol $c_{\text{ap}_N(y), p}^A$ cannot stem from Ω_{LA} , since then p would be of the form $[c_{\text{ap}_N(y), p}^A, +\infty)$ or $[c_{\text{ap}_N(y), p}^A, r_u)$ for some real value r_u and we would have $c_{\text{ap}_N(y), p}^A = d^A$, i.e. $c_{\text{ap}_N(y), p}^A$ and $c_{\text{ap}_N(y), p}^A + \varepsilon$ would be A -covered by $\mathcal{I}_{\text{ap}_N(y), N}$ – a contradiction, as we have already argued then p cannot be a point interval. Hence, there exists a constant symbol $e \in \text{bconsts}(N)$ so that $e \neq d$ and $c_{\text{ap}_N(y), p}^A = \alpha_{e+\varepsilon}$ and thus p is either of the form $(e^A, +\infty)$ or (e^A, r_u) for some real value r_u . In case of $d^A \neq e^A$, the instantiation point $\alpha_{d+\varepsilon}$ results in a partition $p' \in \mathcal{P}_{\text{ap}_N(y), N, A}$ such that $\alpha_{d+\varepsilon}^A \in p'$ and $p' \neq p$. But this contradicts our assumption $\alpha_t^A = \alpha_{e+\varepsilon}^A \in p$. Consequently, $d^A = e^A$. As \widehat{N}_x contains $\text{Ax}_{\{\alpha_{d+\varepsilon}\}, \{e\}}$ as a subset, A must be a hierachic model of the axiom $d = e \rightarrow \alpha_{d+\varepsilon} = \alpha_{e+\varepsilon}$. This yields a contradiction, since above we concluded $\alpha_{d+\varepsilon}^A = \alpha_t^A \neq c_{\text{ap}_N(y), p}^A = \alpha_{e+\varepsilon}^A$.

Case $\mathcal{A}, \beta_\varphi \not\models s \approx s'$ for some free atom $s \approx s' \in \Gamma'$. Hence, s and s' are either variables or constant symbols of the free sort, which means they do not contain subterms of the base sort. Since \mathcal{B} and \mathcal{A} behave identical on free-sort constant symbols and $\beta(u) = \beta_\varphi(u)$ for any variable $u \in V_S$, it must hold $\mathcal{B}, \beta \not\models s \approx s'$.

Case $\mathcal{A}, \beta_\varphi \models s \approx s'$ for some $s \approx s' \in \Delta'$. Analogous to the above case, $\mathcal{B}, \beta \models s \approx s'$ holds.

Case $\mathcal{A}, \beta_\varphi \not\models P(s_1, \dots, s_m)$ for some free atom $P(s_1, \dots, s_m) \in \Gamma'$. This translates to $\langle \mathcal{A}(\beta_\varphi)(s_1), \dots, \mathcal{A}(\beta_\varphi)(s_m) \rangle \notin P^A$.

Every s_i of the free sort is either a constant symbol or a variable. Thus, we have $\mathcal{A}(\beta_\varphi)(s_i) = \mathcal{B}(\beta)(s_i) = \varphi_{[\langle P, i \rangle]}(\mathcal{B}(\beta)(s_i))$, since free-sort constant symbols are interpreted in the same way by \mathcal{A} and \mathcal{B} , and because $\beta_\varphi(u) = \beta(u)$ for every free-sort variable u .

Every s_i that is of the base sort must be a variable. Hence, $\mathcal{A}(\beta_\varphi)(s_i) = c_{[\langle P, i \rangle], p}^{\mathcal{A}} = \varphi_{[\langle P, i \rangle]}(\mathcal{B}(\beta)(s_i))$, where p is the partition in $\mathcal{P}_{[\langle P, i \rangle], N, \mathcal{A}}$ which contains $\beta(s_i)$ (and thus also $\beta_\varphi(s_i)$) and where we have $\text{ap}_N(s_i) = [\langle P, i \rangle]$.

Put together, this yields $\langle \varphi_{[\langle P, 1 \rangle]}(\mathcal{B}(\beta)(s_1)), \dots, \varphi_{[\langle P, m \rangle]}(\mathcal{B}(\beta)(s_m)) \rangle \notin P^{\mathcal{A}}$. But then, by construction of \mathcal{B} , we have $\langle \mathcal{B}(\beta)(s_1), \dots, \mathcal{B}(\beta)(s_m) \rangle \notin P^{\mathcal{B}}$, which entails $\mathcal{B}, \beta \not\models P(s_1, \dots, s_m)$.

Case $\mathcal{A}, \beta_\varphi \models P(s_1, \dots, s_m)$ for some free atom $P(s_1, \dots, s_m) \in \Delta'$. Analogous to the above case we conclude $\mathcal{B}, \beta \models P(s_1, \dots, s_m)$.

Altogether, we have shown $\mathcal{B} \models N$. □

We have already pointed out that we intend to iteratively instantiate base-sort variables in an initial clause set N by means of the construction described in Lemma 13. However, for efficiency reasons it is not desirable to recompute all necessary information such as argument position classes, sets of instantiation points and the like from scratch at each stage. The next lemma shows that the ingredients for the elimination of base-sort variables are invariant under instantiation. The main reason is that when x is instantiated, constraints $x = c$ are introduced to the clause and free atoms $P(\dots, x, \dots)$ remain untouched (however, x might be renamed afterwards). Consequently, it is indeed sufficient to compute, for instance, the relation \Leftarrow_N and the set of instantiation points $\mathcal{I}_{[\langle P, i \rangle], N}$ once and for all at the beginning for every argument position pair $\langle P, i \rangle$ that is of interest and only use it at the appropriate stages of the overall instantiation process. How this can be done will be described in detail in Section 5.

Lemma 14. Let N be a clause set in normal form that contains a clause C in which a base-sort variable x occurs. Further assume $\text{Ax}_{\alpha\text{consts}(N), \text{bconsts}(N)} \subseteq N$ and for every $\alpha_{e+\varepsilon} \in \alpha\text{consts}(N)$ we have $e \in \text{bconsts}(N)$. Suppose \widehat{N}_x is constructed from N as described in Lemma 13 and variables have been renamed so that all clauses in \widehat{N}_x are variable disjoint. We observe the following facts:

- (i) $\Leftarrow_N = \Leftarrow_{\widehat{N}_x}$.
- (ii) $\text{bconsts}(N) = \text{bconsts}(\widehat{N}_x)$ and $\text{fconsts}(N) = \text{fconsts}(\widehat{N}_x)$.
- (iii) For every argument position class $[\langle P, i \rangle]$ induced by \Leftarrow_N we have $\mathcal{I}_{[\langle P, i \rangle], N} = \mathcal{I}_{[\langle P, i \rangle], \widehat{N}_x}$.
- (iv) \widehat{N}_x is in normal form, $\text{Ax}_{\alpha\text{consts}(\widehat{N}_x), \text{bconsts}(\widehat{N}_x)} \subseteq \widehat{N}_x$, and for every $\alpha_{e+\varepsilon} \in \alpha\text{consts}(\widehat{N}_x)$ we have $e \in \text{bconsts}(\widehat{N}_x)$.

Proof. We prove the different parts separately.

(i): The relations \Leftarrow_N and $\Leftarrow_{\widehat{N}_x}$ exclusively depend on the free parts of the clauses in N and \widehat{N}_x , respectively, irrespective of variable names. Since $\mathcal{I}_{\text{ap}_N(x), N}$ is nonempty, all free parts of clauses in N do occur in \widehat{N}_x (modulo variable renaming), and vice versa.

(ii): $\text{bconsts}(N) \subseteq \text{bconsts}(\widehat{N}_x)$ and $\text{fconsts}(N) \subseteq \text{fconsts}(\widehat{N}_x)$ hold due to $\mathcal{I}_{\text{ap}_N(x), N}$ being nonempty. $\text{bconsts}(\widehat{N}_x) \subseteq \text{bconsts}(N)$ is a consequence of $\mathcal{I}_{\text{ap}_N(x), N} \cap \Omega_{\text{LA}} \subseteq \text{bconsts}(N)$. $\text{fconsts}(\widehat{N}_x) \subseteq \text{fconsts}(N)$ is true, because free parts of clauses are merely copied and variables renamed.

(iii): We start with two observations:

- (1) For every clause $D \neq C$ in N , we find a clause \widehat{D} in \widehat{N}_x such that D and \widehat{D} are the same modulo variable renaming. Conversely, every \widehat{D} in \widehat{N}_x is either identical to a clause D in N modulo renaming of variables or it is equal to $C[x/c]$ up to variable renaming for some constant symbol c .
- (2) Regarding C we find the instance $C[x/\alpha_{-\infty}]$ in \widehat{N}_x , possibly with renamed variables. Consequently, for every base-sort variable $y \neq x$ for which C contains an atomic constraint $y \triangleleft s$ and a free atom $Q(\dots, y, \dots)$ with y in the j -th argument position, $C[x/\alpha_{-\infty}]$ contains the constraint $y \triangleleft s$ and the free atom $Q(\dots, y, \dots)$ with y in the j -th argument position – again, modulo variable renaming. The converse also holds.

That said, we distinguish two cases for every argument position pair $\langle P, i \rangle$.

If $\langle P, i \rangle$ does not belong to the equivalence class $\text{ap}_N(x)$, i.e. $\text{ap}_N(x) \neq [\langle P, i \rangle]$, then we get $\mathcal{I}_{[\langle P, i \rangle], N} = \mathcal{I}_{[\langle P, i \rangle], \widehat{N}_x}$, since (1) and (2) together entail $\mathcal{I}_{Q,j,N} = \mathcal{I}_{Q,j,\widehat{N}_x}$ for all $\langle Q, j \rangle \in [\langle P, i \rangle]$.

If $[\langle P, i \rangle] = \text{ap}_N(x)$, then there is an argument position pair $\langle Q, j \rangle \in [\langle P, i \rangle]$ so that C contains a free atom $Q(\dots, x, \dots)$ in which x is the j -th argument. Moreover, \widehat{N}_x is a superset of $\{C[x/c] \mid c \in \mathcal{I}_{\text{ap}_N(x), N}\}$ modulo renaming of variables. This entails $\mathcal{I}_{\text{ap}_N(x), N} \setminus \{\alpha_{-\infty}\} \subseteq \mathcal{I}_{Q,j,\widehat{N}_x}$, and therefore, $\mathcal{I}_{\text{ap}_N(x), N} \subseteq \mathcal{I}_{[\langle P, i \rangle], \widehat{N}_x}$.

On the other hand, assume there is an instantiation point in $\mathcal{I}_{[\langle P, i \rangle], \widehat{N}_x}$ that is not in $\mathcal{I}_{\text{ap}_N(x), N}$, i.e. there is a clause \widehat{D} in \widehat{N}_x of which a variable-renamed variant has not already been in N , and in which an atomic constraint $y \triangleleft s$ and a free atom $Q(\dots, y, \dots)$ occur with y in the j -th argument position. By (1), \widehat{D} must be a variable-renamed variant of $C[x/c]$ for some constant symbol c . But according to (2), we must have $(y \triangleleft s) = (x = c)$ modulo variable renaming. But this constraint can only lead to an instantiation point that is contained in $\mathcal{I}_{\text{ap}_N(x), N}$ – a contradiction. Hence, we also have $\mathcal{I}_{[\langle P, i \rangle], \widehat{N}_x} \subseteq \mathcal{I}_{\text{ap}_N(x), N}$.

Put together, we just derived $\mathcal{I}_{[\langle P, i \rangle], N} = \mathcal{I}_{[\langle P, i \rangle], \widehat{N}_x}$ for this case.

(iv): The construction of \widehat{N}_x from N preserves the normal form property for each clause individually, since we start from a clause in normal form, copy the free part of the clause and no new variables are introduced to the constraint part which do not also occur in the free part. Afterwards, consistent renaming of variables ensures pairwise variable disjointness of the clauses in \widehat{N}_x .

$\text{Ax}_{\alpha\text{consts}(\widehat{N}_x), \text{bconsts}(\widehat{N}_x)} \subseteq \widehat{N}_x$ is a consequence of $\text{Ax}_{\alpha\text{consts}(N), \text{bconsts}(N)} \subseteq N \setminus \{C\}$ (true by assumption and the fact that C 's free part is not empty but the free parts of axioms are) and $\text{Ax}_{\mathcal{I}_{\text{ap}_N(x), N}, \text{bconsts}(N)} \subseteq \widehat{N}_x$ (true by construction). Moreover, $\text{Ax}_{\alpha\text{consts}(\widehat{N}_x), \text{bconsts}(\widehat{N}_x)} = \text{Ax}_{\alpha\text{consts}(N), \text{bconsts}(N)} \cup \text{Ax}_{\mathcal{I}_{\text{ap}_N(x), N}, \text{bconsts}(N)}$ is ensured by (ii) and the fact that $\alpha\text{consts}(\widehat{N}_x) \subseteq \alpha\text{consts}(N) \cup \mathcal{I}_{\text{ap}_N(x), N}$.

Suppose there is a constant symbol $\alpha_{e+\varepsilon}$ in \widehat{N}_x so that e does not occur as constant symbol in \widehat{N}_x . Hence, $\alpha_{e+\varepsilon}$ must have been an instantiation point in $\mathcal{I}_{\text{ap}_N(x), N}$. Due to (ii), e cannot occur in N either. Consequently, $\alpha_{e+\varepsilon}$ has been introduced to $\mathcal{I}_{\text{ap}_N(x), N}$ by the occurrence of an atomic constraint $y = \alpha_{e+\varepsilon}$ in some clause in N . But we assumed $e' \in \text{bconsts}(N)$ for every $\alpha_{e'+\varepsilon} \in \alpha\text{consts}(N)$ – a contradiction. \square

4 Instantiation of Free-Sort Variables

Checking satisfiability of a finite clause set N over the classical Bernays-Schönfinkel fragment can be done naïvely by trying all Herbrand interpretations (we assume N contains at least one constant symbol). One key argument is that the domain of these interpretations is the set of all constant symbols occurring in N and is thus finite. If we add equality to the fragment, the canonical interpretations are Herbrand interpretations modulo congruences on the occurring

constant symbols. Although this means we have to consider more than one domain, there are still only finitely many of them. Moreover, their size is upper bounded by the number of constant symbols in N . This idea of canonical domains can easily be transferred to hierarchic models over the BSR fragment with simple bounds. More precisely, we can prove that any clause set N over the BSR fragment with simple bounds is satisfiable w.r.t. \mathfrak{M} if and only if there exists a hierarchic model \mathcal{A} so that its free domain is the set of free-sort constant symbols in N modulo a congruence relation \sim , i.e. $\mathcal{S}^{\mathcal{A}} = \text{fconsts}(N)/\sim$, and so that every free-sort constant symbol c is interpreted by the corresponding congruence class $[c]_{\sim}$.

Lemma 15. Let N be a clause set in normal form and let \mathcal{A} be a hierarchic model of N . We assume $\text{fconsts}(N)$ to contain at least one constant symbol (otherwise we may add the tautology $\| \rightarrow c \approx c$ to N).

By $\widehat{\mathcal{S}^{\mathcal{A}}}$ we denote the restricted domain $\{a \in \mathcal{S}^{\mathcal{A}} \mid \text{there is a } d \in \text{fconsts}(N) \text{ such that } a = d^{\mathcal{A}}\}$. Let $\sim_{\mathcal{A}}$ be the binary relation on $\text{fconsts}(N)$ satisfying $c \sim_{\mathcal{A}} d$ if and only if $c^{\mathcal{A}} = d^{\mathcal{A}}$.

We can construct a hierarchic model \mathcal{B} of N which meets the following requirements:

- (i) the domain $\mathcal{S}^{\mathcal{B}}$ is the set $\text{fconsts}(N)/\sim_{\mathcal{A}}$ of equivalence classes w.r.t. $\sim_{\mathcal{A}}$;
- (ii) $d^{\mathcal{B}} = [d]_{\sim_{\mathcal{A}}}$ for every free-sort constant symbol $d \in \text{fconsts}(N)$;
- (iii) $c^{\mathcal{B}} = c^{\mathcal{A}}$ for every base-sort constant symbol $c \in \text{bconsts}(N) \cup \alpha\text{consts}(N)$; and
- (iv) for every atom A in N and every variable assignment $\beta : V_S \cup V_R \rightarrow \mathcal{S}^{\mathcal{A}} \cup \mathbb{R}$ for which $\beta(u) \in \widehat{\mathcal{S}^{\mathcal{A}}}$ for all $u \in V_S$, we have $\mathcal{A}, \beta \models A$ if and only if $\mathcal{B}, \beta_{\sim_{\mathcal{A}}} \models A$, where for any variable $v \in V_R \cup V_S$ we set

$$\beta_{\sim_{\mathcal{A}}}(v) := \begin{cases} [d]_{\sim_{\mathcal{A}}} & \text{if } v \in V_S \text{ and } \beta(v) = d^{\mathcal{A}} \\ & \text{for some } d \in \text{fconsts}(N), \\ \beta(v) & \text{if } v \in V_R. \end{cases}$$

Proof. We construct the hierarchic interpretation \mathcal{B} as follows:

- $\mathcal{S}^{\mathcal{B}} := \text{fconsts}(N)/\sim_{\mathcal{A}}$.
- For every constant symbol $d \in \text{fconsts}(N)$, we set $d^{\mathcal{B}} := [d]_{\sim_{\mathcal{A}}}$.
- For every constant symbol $c \in \text{bconsts}(N) \cup \alpha\text{consts}(N)$, we set $c^{\mathcal{B}} := c^{\mathcal{A}}$.
- To help the formulation of the interpretation of predicate symbols under \mathcal{B} we first define the function $\psi : \mathcal{S}^{\mathcal{B}} \cup \mathbb{R} \rightarrow \mathcal{S}^{\mathcal{A}} \cup \mathbb{R}$ by

$$\psi(a) := \begin{cases} d^{\mathcal{A}} & \text{if } a = [d]_{\sim_{\mathcal{A}}} \in \mathcal{S}^{\mathcal{B}} \text{ for some free-sort} \\ & \text{constant symbol } d, \\ a & \text{if } a \in \mathbb{R}. \end{cases}$$

(Please note that this is well-defined, since $d \sim_{\mathcal{A}} d'$ entails $d^{\mathcal{A}} = d'^{\mathcal{A}}$.)

For every predicate symbol P/m that occurs in N and for all arguments a_1, \dots, a_m of appropriate sorts, we define the interpretation of P under \mathcal{B} such that $\langle a_1, \dots, a_m \rangle \in P^{\mathcal{B}}$ if and only if $\langle \psi(a_1), \dots, \psi(a_m) \rangle \in P^{\mathcal{A}}$.

Obviously, requirements (i), (ii) and (iii) are satisfied. Due to the fact that we find a β whose image is a subset of $\widehat{\mathcal{S}^{\mathcal{A}}}$ (as described in (iv)) for every variable assignment $\gamma : V_R \cup V_S \rightarrow \mathcal{S}^{\mathcal{B}} \cup \mathbb{R}$ so that $\beta_{\sim_{\mathcal{A}}} = \gamma$, a proof of (iv) entails $\mathcal{B} \models N$. Hence, it remains to show (iv).

Suppose A is an atomic constraint $s \triangleleft t$. The definitions of \mathcal{B} and $\beta_{\sim_{\mathcal{A}}}$ immediately imply the equivalence of $\mathcal{A}, \beta \models A$ and $\mathcal{B}, \beta_{\sim_{\mathcal{A}}} \models A$.

Suppose A is of the form $s \approx t$, s and t being variables or constant symbols of the free sort, respectively. Because of $\mathcal{A}(\beta)(s), \mathcal{A}(\beta)(t) \in \tilde{\mathcal{S}}^{\mathcal{A}}$, there exist constant symbols $d_s, d_t \in \text{fconsts}(N)$ such that $d_s^{\mathcal{A}} = \mathcal{A}(\beta)(s)$ and $d_t^{\mathcal{A}} = \mathcal{A}(\beta)(t)$. Consequently, $\mathcal{A}(\beta)(s) = \mathcal{A}(\beta)(t)$ holds if and only if $d_s \sim_{\mathcal{A}} d_t$, i.e. if and only if $[d_s]_{\sim_{\mathcal{A}}} = [d_t]_{\sim_{\mathcal{A}}}$. Put differently, $\mathcal{A}, \beta \models s \approx t$ is equivalent to $\mathcal{B}, \beta_{\sim_{\mathcal{A}}} \models s \approx t$.

Suppose A is of the form $P(t_1, \dots, t_m)$ for some predicate symbol P/m occurring in N . As we have already argued in the previous case, we can find for every t_i of the free sort a constant symbol d_i fulfilling $d_i^{\mathcal{A}} = \mathcal{A}(\beta)(t_i)$.

If $t_i = c \in \text{fconsts}(N)$, then $\psi(\mathcal{B}(\beta_{\sim_{\mathcal{A}}})(c)) = \psi([c]_{\sim_{\mathcal{A}}}) = c^{\mathcal{A}} = \mathcal{A}(\beta)(c)$.

If $t_i = u \in V_{\mathcal{S}}$, then $\psi(\mathcal{B}(\beta_{\sim_{\mathcal{A}}})(u)) = \psi(\beta_{\sim_{\mathcal{A}}}(u)) = \psi([d_i]_{\sim_{\mathcal{A}}}) = d_i^{\mathcal{A}} = \beta(u) = \mathcal{A}(\beta)(u)$.

On the other hand, for all t_i of the base sort, we get $\psi(\mathcal{B}(\beta_{\sim_{\mathcal{A}}})(t_i)) = \mathcal{B}(\beta_{\sim_{\mathcal{A}}})(t_i) = \mathcal{A}(\beta)(t_i)$.

Altogether, this leads to $\langle \psi(\mathcal{B}(\beta_{\sim_{\mathcal{A}}})(t_1)), \dots, \psi(\mathcal{B}(\beta_{\sim_{\mathcal{A}}})(t_m)) \rangle = \langle \mathcal{A}(\beta)(t_1), \dots, \mathcal{A}(\beta)(t_m) \rangle$ and, consequently, also to $\langle \mathcal{B}(\beta_{\sim_{\mathcal{A}}})(t_1), \dots, \mathcal{B}(\beta_{\sim_{\mathcal{A}}})(t_m) \rangle \in P^{\mathcal{B}}$ if and only if $\langle \mathcal{A}(\beta)(t_1), \dots, \mathcal{A}(\beta)(t_m) \rangle \in P^{\mathcal{A}}$.

□

Having such canonical models, it is easy to argue that we can eliminate free-sort variables by exhaustive instantiation with all occurring free-sort constant symbols.

Corollary 16. Let N be a clause set in normal form that contains at least one constant symbol of the free sort. Suppose there is a clause C in N which contains a free-sort variable u . Let the clause set \tilde{N}_u be constructed as follows: $\tilde{N}_u := (N \setminus \{C\}) \cup \{C[u/c] \mid c \in \text{fconsts}(N)\}$. The original clause set N is satisfiable if and only if \tilde{N}_u is satisfiable.

Proof. While the “only if”-part holds because of \tilde{N}_u containing only instances of clauses in N , the “if”-part is slightly more complicated. Let \mathcal{A} be a hierachic model of \tilde{N}_u so that $\mathcal{S}^{\mathcal{A}} = \text{fconsts}(\tilde{N}_u)/\sim$ for some congruence relation \sim on $\text{fconsts}(N)$, and for every free-sort constant symbol c we have $c^{\mathcal{A}} = [c]_{\sim}$. But then $\mathcal{A} \models \{C[u/c] \mid c \in \text{fconsts}(N)\}$ entails $\mathcal{A} \models C$, since the set $\{[c]_{\sim} \mid c \in \text{fconsts}(N)\}$ covers the whole domain $\mathcal{S}^{\mathcal{A}}$ due to $\text{fconsts}(\tilde{N}_u) = \text{fconsts}(N)$. Hence, we obtain $\mathcal{A} \models N$. □

5 The Complexity of Deciding Satisfiability

In the last two sections we have seen how to eliminate base-sort and free-sort variables by means of finite instantiation. In this section, we put these instantiation mechanisms together to obtain a nondeterministic algorithm that decides the hierachic satisfiability problem for the BSR fragment with simple bounds and investigate its complexity. As a measure of the length of clause sets, clauses, atoms and multisets thereof, we use the number of occurrences of constant symbols and variables in the respective object, and denote it by $\text{len}(\cdot)$.

Theorem 17. Let N be a clause set (of length at least 2) that does not contain any constant symbol from $\Omega_{\text{LA}}^{\alpha}$ but at least one free-sort constant symbol. Satisfiability of N w.r.t. \mathfrak{M} can be decided in nondeterministic exponential time. To put it more precisely: the problem lies in $\text{NTIME}(\text{len}(N)^{\mathbf{c} \cdot \text{len}(N)})$ for some constant $\mathbf{c} > 1$.

Proof. We devise a naïve algorithm that decides a given problem instance as follows. As input we assume a finite clause set N such that $\alpha\text{consts}(N) = \emptyset$ and $\beta\text{consts}(N) \neq \emptyset$.

- (I) Transform the input clause set N into normal form by applying three steps to every clause $C = \Lambda \parallel \Gamma \rightarrow \Delta$ in N that contains exactly $k > 0$ distinct base-sort variables x_1, \dots, x_k occurring in Λ but not in $\Gamma \rightarrow \Delta$:

- (I.I) Let \bar{y} be the vector of all variables x_j among x_1, \dots, x_k for which there is an atomic constraint $x_j = c_j$ in Λ , and let \bar{c} denote the corresponding vector of constant symbols c_j . We replace C by $\Lambda[\bar{y}/\bar{c}] \parallel \Gamma \rightarrow \Delta$ in N .
- (I.II) If Λ contains an atomic constraint of the form $x_j \neq c_j$ after Step (I.I), then replace C in N by two clauses $\Lambda', x_j < c_j \parallel \Gamma \rightarrow \Delta$ and $\Lambda', x_j > c_j \parallel \Gamma \rightarrow \Delta$, where $\Lambda' := \Lambda \setminus \{x_j \neq c_j\}$. Iterate this procedure on the newly added clauses until all atomic constraints concerning one of the variables x_1, \dots, x_k have the form $x_j \triangleleft c_j$ with $\triangleleft \in \{<, \leq, \geq, >\}$.
- (I.III) Apply Fourier-Motzkin quantifier elimination to every base-sort variable among x_1, \dots, x_k that is left in N .

We call the resulting clause set N' . In addition, rename variables so that all clauses in N' are variable disjoint.

- (II) Compute the equivalence classes with respect to the equivalence relation $\equiv_{N'}$ induced by N' .

- (III) Compute the set $bconsts(N')$ of all base-sort constant symbols that occur in N' .

For every equivalence class $[\langle P, i \rangle]_{\equiv_{N'}}$ with $P : \xi_1 \times \dots \times \xi_m \in \Pi$ such that $\xi_i = \mathcal{R}$, collect all instantiation points that are connected to it, i.e. compute $\mathcal{I}_{[\langle P, i \rangle], N'}$. Whenever a constant symbol α_t freshly enters the collection of instantiation points, construct the axiom set $Ax_{\{\alpha_t\}, bconsts(N')}$ according to Definition 9 and add all these axioms to a maintained axiom set Ax .

- (IV) Compute the set $fconsts(N')$ of all free-sort constant symbols occurring in N' .

- (V) Perform an all-at-once instantiation process on N' that leads to

$$N'' := \left\{ C[\bar{x}, \bar{u}/\bar{c}, \bar{d}] \mid C \in N' \text{ and } \{x_1, \dots, x_k\} = \text{vars}(C) \cap V_{\mathcal{R}} \text{ and} \right. \\ \left. \{u_1, \dots, u_\ell\} = \text{vars}(C) \cap V_S \text{ and} \right. \\ \left. c_i \in \mathcal{I}_{ap_{N'}(x_i), N'} \text{ and } d_j \in fconsts(N') \text{ for all } i \text{ and } j \right\}.$$

- (VI) Nondeterministically construct a hierachic interpretation \mathcal{A} in three steps:

- (VI.I) Nondeterministically choose a total ordering $\preceq_{\mathcal{A}}$ on the constant symbols in $bconsts(N'')$ $\cup \alphaconsts(N'')$ so that for all $c, d \in bconsts(N'') \cap \mathbb{R}$ it holds $c \preceq_{\mathcal{A}} d$ if and only if $c \leq d$. Based on this ordering, construct (in a deterministic fashion) a mapping $\mu_{\mathcal{A}} : bconsts(N'') \cup \alphaconsts(N'') \rightarrow \mathbb{R}$ so that

- for all $c \in bconsts(N'') \cap \mathbb{R}$ we get $\mu_{\mathcal{A}}(c) = c$, and
- for all $d_1, d_2 \in bconsts(N'') \cup \alphaconsts(N'')$ it holds $\mu_{\mathcal{A}}(d_1) \leq \mu_{\mathcal{A}}(d_2)$ if and only if $d_1 \preceq_{\mathcal{A}} d_2$.

Set $c^{\mathcal{A}} := \mu_{\mathcal{A}}(c)$ for every $c \in bconsts(N'')$.

- (VI.II) Nondeterministically choose a mapping $\nu_{\mathcal{A}} : fconsts(N'') \rightarrow fconsts(N'')$.

While $\nu_{\mathcal{A}}$'s image shall induce the free domain $\mathcal{S}^{\mathcal{A}} := \{\nu_{\mathcal{A}}(c) \mid c \in fconsts(N'')\}$, the value $\nu_{\mathcal{A}}(c)$ shall be assigned to c under \mathcal{A} for every $c \in fconsts(N'')$.

- (VI.III) Let $At(N'')$ denote the set of nonequational ground atoms induced by the essentially ground clause set N'' , formally

$$At(N'') := \{ A[\bar{x}/\bar{c}] \mid \text{there is a nonequational free atom } A \text{ in the free part } \Gamma \rightarrow \Delta \\ \text{ of a clause } (\Lambda, x_1 = c_1, \dots, x_k = c_k \parallel \Gamma \rightarrow \Delta) \in N'' \\ \text{ with } \{x_1, \dots, x_k\} = \text{vars}(A) \}.$$

Given $\text{At}(N'')$, we construct the set $\widehat{\text{At}}(N'')$ by syntactically replacing every base-sort constant symbol c in $\text{At}(N'')$ by $\mu_{\mathcal{A}}(c)$ and every free-sort constant symbol e by $\nu_{\mathcal{A}}(e)$. Nondeterministically choose a subset $\widehat{\text{At}}_{\mathcal{A}}(N'')$ of $\widehat{\text{At}}(N'')$ that represents the atoms in $\widehat{\text{At}}(N'')$ which shall be true under \mathcal{A} , i.e. construct a Herbrand model over the atoms in $\widehat{\text{At}}(N'')$.

(VII) Check whether \mathcal{A} is a hierachic model of $N'' \cup \text{Ax}$. \diamond

Regarding the correctness of the algorithm, there are two crucial points which we need to address, namely (a) the equisatisfiability of N and N' , and (b) the equisatisfiability of N' and $N'' \cup \text{Ax}$.

Ad (a): It is straightforward to check that (I.I) and (I.II) lead to equisatisfiable clause sets with respect to the standard semantics of linear arithmetic.

The Fourier-Motzkin elimination step in (I.III) works for existentially quantified real-valued variables (cf. [DE73], [KS08]). Given a clause $C = \Lambda \parallel \Gamma \rightarrow \Delta$ that contains a base-sort variable x occurring in Λ but not in $\Gamma \rightarrow \Delta$, let \bar{z} denote the vector of all variables in Λ except for x . Recall that C can be informally understood as $\forall \bar{z} \forall x ((\bigwedge \Lambda \wedge \bigwedge \Gamma) \rightarrow \bigvee \Delta)$. Since x does neither occur in Δ nor in Γ and since any formula $\phi \rightarrow \psi$ is equivalent to $\neg\phi \vee \psi$, we can equivalently write $\forall \bar{z} ((\exists x \wedge \Lambda) \wedge \bigwedge \Gamma) \rightarrow \bigvee \Delta$. Now Fourier-Motzkin variable elimination can be applied to the part $\exists x \wedge \Lambda$ to transform it into a formula $\bigwedge \Lambda'$ over linear arithmetic constraints (each of the form $c \triangleleft d$ or $y \triangleleft e$) so that $\bigwedge \Lambda'$ is equivalent to $\exists x \wedge \Lambda$ with respect to the standard semantics of linear arithmetic, Λ' does not contain x anymore and all variables and constant symbols in Λ' have already occurred in Λ (cf. [DE73]).

Treating all base-sort variables that occur in Λ but not in $\Gamma \rightarrow \Delta$ as described above, we finally obtain a clause $\Lambda'' \parallel \Gamma \rightarrow \Delta$ that is equivalent to $\Lambda \parallel \Gamma \rightarrow \Delta$ and is in normal form.

Ad (b): The construction of clause set N'' done in Step (V) resembles $|\text{vars}(N') \cap V_{\mathcal{R}}|$ consecutive applications of Lemma 13 to the base-sort variables x_1, \dots, x_k followed by $|\text{vars}(N') \cap V_{\mathcal{S}}|$ consecutive applications of Corollary 16 to the free-sort variables u_1, \dots, u_{ℓ} in order to instantiate all variables that occur in N' .

For an arbitrary clause C in N' we have $\text{fconsts}(C[\bar{x}/\bar{c}] [u_1/d_1] \dots [u_j/d_j]) \subseteq \text{fconsts}(N')$ for every j , $0 \leq j \leq \ell$. Hence, Corollary 16 entails equisatisfiability of $N'' \cup \text{Ax}$ and the intermediate set $N''' \cup \text{Ax}$ where $N''' := \{C[\bar{x}/\bar{c}] \mid C \in N' \text{ and } \{x_1, \dots, x_k\} = \text{vars}(C) \cap V_{\mathcal{R}} \text{ and } c_i \in \mathcal{I}_{\text{ap}_{N'}(x_i), N'} \text{ for all } i\}$.

It remains to show equisatisfiability of N' and $N''' \cup \text{Ax}$. In order to do so, we first reduce the problem to showing equisatisfiability of $N' \cup \text{Ax}$ and $N''' \cup \text{Ax}$: If N' is satisfiable, then so is $N' \cup \text{Ax}$ by Proposition 10, since N' does not contain constant symbols from $\Omega_{\text{LA}}^{\alpha}$. Conversely, N' is obviously satisfiable whenever $N' \cup \text{Ax}$ is.

Let x_1, \dots, x_k be a list of pairwise distinct base-sort variables such that $\{x_1, \dots, x_k\} := \text{vars}(N') \cap V_{\mathcal{R}} = \text{vars}(N' \cup \text{Ax}) \cap V_{\mathcal{R}}$. We set $N'_0 := N'$ and for every j , $1 \leq j \leq k$, we define

$$N'_j := \{C[x_j/c_j] \mid C \in N'_{j-1} \text{ and } c_j \in \mathcal{I}_{\text{ap}_{N'}(x_j), N'}\}.$$

As the x_1, \dots, x_k are pairwise distinct and since the involved substitution operations only substitute variables with constant symbols, we can equivalently write

$$N'_j = \{C[x_1/c_1] \dots [x_j/c_j] \mid C \in N' \text{ and } c_i \in \mathcal{I}_{\text{ap}_{N'}(x_i), N'} \text{ and } 1 \leq i \leq j\}$$

for every j , $1 \leq j \leq k$. Variable disjointness of the clauses in N' entails $N'_k = N'''$, since in this case the iterative substitution in the construction of N'_k yields the same result as simultaneous substitution in the construction of N''' does.

Consider the sequence N'_0, N'_1, \dots, N'_k of clause sets, for which we know $N'_0 = N'$ and $N'_k = N'''$. If we rename the variables in each set in the sequence such that the clauses are variable disjoint, we can apply Lemma 13 k times to conclude equisatisfiability of $N'_0 \cup \text{Ax}$, $N'_1 \cup \text{Ax}$ and so on

up to $N'_k \cup \text{Ax}$. To support this claim, we need to show that the prerequisites of Lemma 13 are fulfilled along the sequence. They certainly are for the starting point $N'_0 \cup \text{Ax}$, since we assume N' to be in normal form and to not contain constant symbols α_t . For the rest of the sequence, we invoke Lemma 14, since it ensures $\Leftarrow_{N'_0 \cup \text{Ax}} = \dots = \Leftarrow_{N'_k \cup \text{Ax}}$ and $\mathcal{I}_{[\langle P, i \rangle], N'_0} = \dots = \mathcal{I}_{[\langle P, i \rangle], N'_k}$ for all equivalence classes $[\langle P, i \rangle]$ induced by $\Leftarrow_{N'_0 \cup \text{Ax}}$.

The following observations justify why it is legitimate for the algorithm described in steps (I) to (VII) to mostly ignore the axioms in Ax .

- (b.1) $\Leftarrow_{N'} = \Leftarrow_{N' \cup \text{Ax}}$, because the relation $\Leftarrow_{N' \cup \text{Ax}}$ exclusively depends on the free parts of clauses in $N' \cup \text{Ax}$, but the clauses in Ax do not have free parts.
- (b.2) $\text{bconsts}(N') = \text{bconsts}(N' \cup \text{Ax})$, since the definition of Ax entails $\text{bconsts}(\text{Ax}) \subseteq \text{bconsts}(N')$.
- (b.3) $\text{fconsts}(N') = \text{fconsts}(N' \cup \text{Ax})$ holds because clauses in Ax do not contain constant symbols of the free sort.
- (b.4) $\mathcal{I}_{[\langle P, i \rangle], N'} = \mathcal{I}_{[\langle P, i \rangle], N' \cup \text{Ax}}$, since only atomic constraints that involve one base-sort variable contribute to the set of instantiation points, but Ax is a set of ground clauses.

Altogether, the iterative application of Lemma 13 shows that $N' \cup \text{Ax}$ has a hierarchic model if and only if $N''' \cup \text{Ax}$ has one. But as we have already argued above, this entails the equisatisfiability of N' and $N''' \cup \text{Ax}$ and even $N'' \cup \text{Ax}$.

This finishes our considerations regarding the correctness of the presented algorithm. Next, we investigate its running time. In order to do so, we take a look at every step individually. (I): While the substitution operations in Step (I.I) take a total amount of time that is polynomial in $\text{len}(N)$, the length of the clause set does not grow. The second step, however, may blow up the length of the clause set exponentially. Every clause $\Lambda \parallel \Gamma \rightarrow \Delta$ might be copied up to $2^{|\Lambda|}$ times, since Λ can contain at most $|\Lambda|$ constraints of the form $x_j \neq c_j$. Hence, Step (I.II) increases the length of the clause set to not more than $\text{len}(N) \cdot 2^{\text{len}(N)}$ and in the worst case takes time polynomial in that new length. Given a multiset Λ of atomic constraints, in which variables $x_1, \dots, x_k \in \text{vars}(\Lambda)$ are supposed to be eliminated one after another by the Fourier-Motzkin procedure, we can partition Λ into $k+1$ parts $\Lambda_0, \Lambda_1, \dots, \Lambda_k$ so that for all $i > 0$ the part Λ_i contains all atomic constraints that involve x_i and only those, and $\Lambda_0 := \Lambda \setminus (\Lambda_1 \cup \dots \cup \Lambda_k)$. It turns out that eliminating a variable x_i from Λ results in a multiset $(\Lambda \setminus \Lambda_i) \cup \widehat{\Lambda}_i$ where $|\widehat{\Lambda}_i|$ is at most $|\Lambda_i|^2$. Hence, after eliminating all x_i , we end up with a multiset of atomic constraints of size $|\Lambda_0| + |\Lambda_1|^2 + \dots + |\Lambda_k|^2 \leq |\Lambda|^2$. Consequently, Step (I.II) may increase the length of the clause set at most quadratically. The time taken for this step is polynomial in that new length. Overall, we end up with a length of at most $\text{len}(N)^2 \cdot 2^{2 \cdot \text{len}(N)} \leq \text{len}(N)^{3 \cdot \text{len}(N)}$ for N' (recall that we assumed $\text{len}(N) \geq 2$).

(II): This step can be performed in time that is polynomial in the length of N' using an efficient union-find data structure.

(III): The computation of $\text{bconsts}(N')$ and the collection of all relevant instantiation points takes time polynomial in the length of N' . The set of instantiation points for any base-sort variable is a subset of $\text{bconsts}(N') \cup \{\alpha_{d+\varepsilon} \mid d \in \text{bconsts}(N')\} \cup \{\alpha_{-\infty}\}$. It is worthwhile to note that Steps (I.I) to (I.III) do not lead to a change in the number of instantiation points, since they only modify atomic constraints that do not contribute to instantiation points. The reason is that the variables x_1, \dots, x_k addressed in Step (I) do not occur in the free parts of the modified clause. For every argument position pair $\langle P, i \rangle$, there are at most $\text{len}(N)$ instantiation points, as every atomic constraint $y \triangleleft c$ in N can induce at most one instantiation point (either c or $\alpha_{c+\varepsilon}$). To account for $\alpha_{-\infty}$: if there is a base-sort variable y to be instantiated in N' at all, then N' must also contain a free atom $Q(\dots, y, \dots)$ which did already occur in N , and which thus also contributes to the length of N . In addition, we have $\text{bconsts}(N') \subseteq \text{bconsts}(N)$, leading to $|\text{bconsts}(N')| \leq \text{len}(N)$. The construction of the required axiom set Ax can be done in polynomial time in $\text{len}(N') + \text{len}(\text{Ax})$, where we can bound the length of the axiom set from above by $2 \cdot 2 \cdot \text{len}(N) + 2 \cdot 4 \cdot \text{len}(N)^2 \leq 10 \cdot \text{len}(N)^2$.

(IV): The extraction of $\text{fconsts}(N')$ does not take longer than polynomial time in the length of N' .

(V): At first, we consider each clause $C = \Lambda \parallel \Gamma \rightarrow \Delta$ in N' separately. Since $|\text{fconsts}(N')|$ is upper bounded by $\text{len}(N)$ (every free-sort constant symbol in N' did already occur in N), instantiation of the free-sort variables yields a factor of at most $\text{len}(N)^{|\text{vars}(\Gamma \rightarrow \Delta) \cap V_s|}$. We have already argued – when looking at Step (III) – that the number of instantiations points for each base-sort variable is bounded from above by $\text{len}(N)$. Hence, instantiation of all base-sort variables in the clause adds a factor of at most $\text{len}(N)^{|\text{vars}(\Gamma \rightarrow \Delta) \cap V_R|}$. There are at most $\text{len}(\Gamma \rightarrow \Delta)$ different variables in C that need to be instantiated, and as $\Gamma \rightarrow \Delta$ did already occur in N (modulo variable renaming), we have $\text{len}(\Gamma \rightarrow \Delta) \leq \text{len}(N)$. When instantiating a clause, the constraint part may increase in length, namely by at most double the number of instantiated variables. In the worst case, we thus get triple the length of the original, e.g. in case of instantiating the clause $\parallel \rightarrow P(x)$ with the default instantiation point α_∞ we obtain $x = \alpha_\infty \parallel \rightarrow P(x)$. In total, instantiating a single clause $C = \Lambda \parallel \Gamma \rightarrow \Delta$ taken from N' leads to a clause set of length at most $3 \cdot \text{len}(C) \cdot \text{len}(N)^{\text{len}(\Gamma \rightarrow \Delta)}$. Consequently, we can upper bound the length of the fully instantiated clause set N'' by $3 \cdot \text{len}(N') \cdot \text{len}(N)^{\text{len}(N)} \leq 3 \cdot \text{len}(N)^{4 \cdot \text{len}(N)}$. Instantiating the set of clauses needs only time that is bounded by some polynomial in $\text{len}(N'')$.

(VI): The construction of \mathcal{A} can be done nondeterministically in time that is bounded from above by some polynomial in the length of N'' .

(VII): The check whether \mathcal{A} satisfies $N'' \cup \text{Ax}$ can be performed in a deterministic fashion in time polynomial in the length of $N'' \cup \text{Ax}$.

Taking all the above results into account, we can upper bound the running time of the algorithm by some polynomial in $\text{len}(N)^{4 \cdot \text{len}(N)}$. Hence, there is some constant $c \geq 4$ such that the nondeterministic running time lies in $\mathcal{O}(\text{len}(N)^{c \cdot \text{len}(N)})$. Consequently, the problem of deciding whether a finite clause set N is satisfiable, lies in NEXPTIME. \square

NEXPTIME-completeness of satisfiability for the Bernays-Schönfinkel-Ramsey fragment of first-order logic without equality (cf. [Lew80]) immediately yields NEXPTIME-hardness of satisfiability for finite clause sets over the BSR fragment with simple bounds. Together with Theorem 17 we thus obtain NEXPTIME-completeness of the problem.

Corollary 18. The problem of deciding satisfiability w.r.t. \mathfrak{M} of finite clause sets over the BSR fragment with simple bounds is NEXPTIME-complete.

6 Beyond Simple Bounds

In this section we long to answer the question how simple our constraints have to be. The most complex atomic constraints we have allowed by now are of the form $x \triangleleft c$. Being able to cope with this kind, we can leverage the idea of *flattening* to deal with more complicated constraints such as $3x + c < 1$. The basic idea rests on two steps:

- (I) Transform this constraint into the equivalent $x < 1 - \frac{1}{3}c$.
- (II) Introduce a fresh Skolem constant b , transform the constraint into $x < b$ and add a *defining clause* C_b expressing $b = 1 - \frac{1}{3}c$ to the clause set.

These two steps already indicate that this technique is restricted to atomic constraints that are either ground or univariate and linear, and in which the standard operations addition, subtraction, multiplication and division on the reals may be involved. But we may even allow free function symbols $g : \xi_1 \times \dots \times \xi_m \rightarrow \xi$ with $\xi_1, \dots, \xi_m, \xi \in \{\mathcal{R}, \mathcal{S}\}$, as long as all subterms occurring below g are ground.

The key insight at this point is that complex ground terms s can be replaced by fresh constant symbols b_s of the corresponding sort, if we add a *defining clause* C_{b_s} that identifies b_s with s (a

technique called *basification* in [KW12]) – see below for details. Of course, the replacement of s must be done consistently throughout the clause set. Consequently, we can extend the syntax of clauses so that more complex terms are admitted.

Definition 19 (BSR Clause Fragment with Ground LA Bounds). Let Ω' collect nonconstant free function symbols equipped with sorting information. An *atomic constraint* is of the form $s \triangleleft s'$ with $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$ and well-sorted base-sort terms s, s' over symbols in $\Omega_{\text{LA}} \cup \{+, -, \cdot, /\} \cup \Omega \cup \Omega'$ in which every subterm $g(s_1, \dots, s_m)$ with $g \in \Omega'$ is ground and where $s \triangleleft s'$ contains at most one occurrence of a base-sort variable and no free-sort variables. A *free atom* A is either of the form $s \approx s'$ with s, s' being either free-sort variables or well-sorted ground terms of the free sort over function symbols in $\Omega_{\text{LA}} \cup \{+, -, \cdot, /\} \cup \Omega \cup \Omega'$, or A is of the form $P(s_1, \dots, s_m)$, where $P : \xi_1 \times \dots \times \xi_m \in \Pi$ and for each $i \leq m$ the term s_i is of sort ξ_i . If $\xi_i = \mathcal{R}$, then s_i is a base-sort variable, and if $\xi_i = \mathcal{S}$, then s_i is either a free-sort variable or a ground free-sort term over function symbols in $\Omega_{\text{LA}} \cup \{+, -, \cdot, /\} \cup \Omega \cup \Omega'$.

As we have already sketched, we can easily transform a clause set N that contains clauses over the extended syntax into an equisatisfiable set $N_1 \cup N_2$ so that N_1 contains clauses according to Definition 1 only and all clauses in N_2 are of the form $b_s \neq s \parallel \rightarrow \square$ or $\parallel \rightarrow b_s \approx s$. Clearly, we can proceed with N_1 according to Steps (I) to (V) of the algorithm given in the proof of Theorem 17 and thus obtain the essentially ground clause set N''_1 that is equisatisfiable to N_1 and a set Ax of axioms. The construction of a hierachic interpretation \mathcal{A} for N''_1 in Step (VI) can be modified so that it results in an interpretation that also covers all function symbols that occur in N_2 but not in N_1 . Checking whether \mathcal{A} is a hierachic model of $N''_1 \cup \text{Ax} \cup N_2$ can be done easily. Consequently, the hierachic satisfiability problem for clause sets over the extension of the BSR fragment with ground LA bounds is decidable, too.

7 Undecidable Fragments

So far we have described decidable fragments of first-order logic modulo linear arithmetic. In [FW12] it has been shown that already the Bernays-Schönfinkel-Horn (BSH) fragment with addition and subtraction on the reals is undecidable. In the current section we describe more fine-grained undecidable fragments. As it turns out, two-counter machines – of which the halting problem has been proven to be undecidable in [Min67] – can be encoded exclusively using very restricted syntax on the constraint part, such as difference constraints $x - y \triangleleft c$, additive constraints $x + y \triangleleft c$, quotient constraints $x \triangleleft c \cdot y$ (which could equivalently be written $\frac{x}{y} \triangleleft c$, hence the name) or multiplicative constraints $x \cdot y \triangleleft c$. Even more restrictive, in the case of difference constraints and quotient constraints only a single base-sort constant symbol is necessary. In case of quotient and multiplicative constraints, lower and upper bounds on the used variables do not lead to a decidable fragment – which would be the case if we were using variables over the integers.

Difference constraints We use the predicate symbol $M : \mathcal{S} \times \mathcal{R} \times \mathcal{R} \times \mathcal{R}$ to address the state of the machine as follows: $M(u, x, y, z)$ stands for a machine at instruction u with counter values $i_1 = x - z - 1$ and $i_2 = y - z - 1$, where the last argument z keeps track of an offset relative to which x and y store the values of the counters. Following this principle, the increment instruction for the first counter i_1 is encoded by the clause $x' - x = 1 \parallel M(b, x, y, z) \rightarrow M(b', x', y, z)$, which leaves the offset untouched. The offset is an appropriate tool that allows us to have a uniform syntactic structure for all atomic constraints. It is due to the offset encoding that we can easily use a difference constraint when checking whether a counter is zero or not. The conditional decrement instruction is split up in two clauses: the zero case $x - z = 1 \parallel M(b, x, y, z) \rightarrow M(b', x, y, z)$ and the non-zero case $x - z > 1, y' - y = 1, z' - z = 1 \parallel M(b, x, y, z) \rightarrow M(b', x', y, z')$. Hence, by undecidability of the halting problem for two-counter machines, we may conclude that satisfiability for the BSH fragment with difference constraints (requiring only the constant 1 besides the input) and a single free 4-ary predicate symbol is undecidable, too.

Quotient constraints Encoding two-counter machines in the BSH fragment with quotient constraints works very similar. We only need to change the representation of counter values in a state $M(u, x, y, z)$ as follows: $i_1 = -\log_2(\frac{2x}{z}) = -\log_2(x) + \log_2(z) - 1$ and $i_2 = -\log_2(\frac{2y}{z})$. Incrementing the first counter is encoded by $2 \cdot x' = x \parallel M(b, x, y, z) \rightarrow M(b', x', y, z)$, and the conditional decrement instruction is represented by $2 \cdot x = z \parallel M(b, x, y, z) \rightarrow M(b', x, y, z)$ and $2 \cdot x < z, 2 \cdot y' = y, 2 \cdot z' = z \parallel M(b, x, y, z) \rightarrow M(b', x, y', z')$. Analogous to the case of difference constraints, we may now show undecidability of the satisfiability problem for the BSH fragment with quotient constraints (requiring only the constant 2) and a single free 4-ary predicate symbol is undecidable. We have chosen negative exponents for the encoding of the counter values, since this guarantees that the range of the base-sort variables is bounded from below and above. Thus, we could restrict all base-sort variables to values within $(0, 1]$ (by adding appropriate atomic constraints to every clause), and still end up with an undecidable satisfiability problem.

Additive constraints Having additive constraints of the form $x + y \triangleleft c$ at hand, we can simulate subtraction by defining the additive inverse using a constraint $x + x_- = 0$. To keep track of inverses, we adjust the arity of M accordingly. Counter values are represented in the same way as we have done for difference constraints. The increment instruction for the first counter is thus encoded by $x' + x_- = 1, x' + x'_- = 0 \parallel M(b, x, x_-, y, y_-, z, z_-) \rightarrow M(b', x', x'_-, y, y_-, z, z_-)$. It is now straightforward to come up with the encoding of the conditional decrement. Hence, satisfiability of the BSH fragment with additive constraints and a single free predicate symbol of arity 7 is undecidable. However, this time we need two constants, namely 1 and 0.

Multiplicative constraints In order to complete the picture, we shortly leave the realm of linear arithmetic and consider multiplicative constraints of the form $x \cdot y \triangleleft c$. These relate to quotient constraints as additive constraints relate to difference constraints. Hence, combining the previously used ideas of offsets and inverses, we can encode two-counter machines also with multiplicative constraints:

$x \cdot x'_{-1} = 2, x' \cdot x'_{-1} = 1 \parallel M(b, x, x_{-1}, y, y_{-1}, z, z_{-1}) \rightarrow M(b', x', x'_{-1}, y, y_{-1}, z, z_{-1})$ encodes the increment instruction on the first counter, for instance. As in the case of quotient constraints, we could restrict the range of base-sort variables to $(0, 1]$ by suitable constraints. Consequently, this leads to another fragment of which the satisfiability problem is undecidable.

8 Conclusion and Future Work

Our main contribution is a proof showing that satisfiability of the BSR fragment equipped with simple bounds on real arguments is decidable. The key argument describes a satisfiability preserving replacement of universally quantified clauses by finitely many instances which are in essence variable free. The analysis of a naive decision procedure proves that satisfiability of this fragment is NEXPTIME-complete. The complexity result is of particular interest, since satisfiability of the Bernays-Schönfinkel fragment has already been known to be NEXPTIME-complete for more than three decades [Lew80].

Our approach to proving decidability has been tailored to be well-suited for an integration into reasoning procedures that perform efficiently on practically relevant problem instances. This in particular applies to the way we handle the real-sorted part of clauses. For the free parts we could have adopted more fine-grained techniques inspired by the ones we used for the real-sorted part. However, we did not do so for the sake of simplicity and brevity.

It turned out that we leave the realm of decidability as soon as we add elementary operations \circ on the reals to our constraint language, even if we restrict the free part to Horn clauses over the Bernays-Schönfinkel fragment. Constraints of the form $x \circ y \triangleleft c$ are sufficient to obtain an undecidable fragment, where \circ can stand for addition, subtraction, multiplication and division, \triangleleft represents standard relations $<, \leq, =, \neq, \geq, >$, and c is a real-valued constant. This observation nicely complements our main result, since it quite clearly highlights the limits of decidability in

this context. Moreover, it reveals some interplay between real-sorted constraints and the free first-order part. For instance, difference logic (boolean combinations of propositional variables and existentially quantified constraints $x - y \triangleleft c$ with c being a real-valued constant and $\triangleleft \in \{<, \leq\}$) is known to be decidable [MNAM02]. However, we have seen in Section 7 that its combination with the Bernays-Schönfinkel-Horn fragment is sufficient to formalize two-counter machines.

Although the obvious options for further extending the constraint language lead to undecidability there might still be room for improvement. We leave it as future work to investigate the BSR fragment with simple bounds plus constraints of the form $x \triangleleft y$ with x, y being real-valued variables and $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$. On the other hand, it is conceivable to combine other decidable free first-order fragments with simple bounds, preferably ones satisfying the finite model property such as the monadic fragment. As we have already pointed out, a natural next step for us will be to devise useful decision procedures for the BSR fragment with simple bounds that perform well in practice.

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